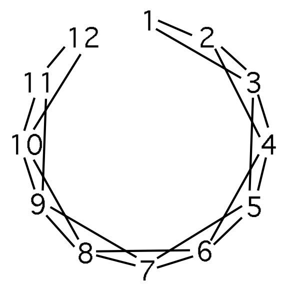
JUNIOR SOLUTIONS 1996

1. Find an arrangement of the numbers from 1 to 12 around the outside of a circle so that the difference between any two numbers next to each other is at most 2.

   Solution.
   The condition that the difference between any two numbers next to each other is at most 2 can be present graphically by drawing a line between the two numbers which are permitted to lie next to each other, as follows.

   From this it can be seen that 1 can only be adjacent to 2 and 3, and then the neighbours of 2 and 3 are determined, and so on. Finally, there are only two ways to arrange the numbers from 1 to 12 satisfying the condition, one being the reverse of the other. Around the circle, the order is 1 2 4 6 8 10 12 11 9 7 5 3 or 1 3 5 7 9 11 12 10 8 6 4 2.

2. At a school of 800 students, 60% attended the school camp. Of these, 70% brought a sleeping bag. Everybody who did not bring a sleeping bag slept in the main hut. Exactly 100 students at the camp did not sleep in the main hut. How many of the students with a sleeping bag slept in the main hut?

   Solution.
   $0.6 \times 800 = 480$ students went for the school camp.

   $0.7 \times 480 = 336$ is the number of the camping students with a sleeping bag. Exactly 100 of these students did not sleep in the main hut so

$336 - 100 = 236$ is the number of the students with a sleeping bag who slept in the main hut.

3. From when he started work, Xavier's pay, which is a whole number of dollars each day, has been worked out by the following rule. If the number of dollars he earned yesterday has 3 as a factor, then today he earns $\frac{2}{3}$ of that number of dollars. If 3 is not a factor of the number earned yesterday, then his pay today is twice yesterday's pay, plus one dollar. His pay on Friday 26 April was 60 dollars.

   (a) What was his pay on Tuesday 23 April?

   (b) He claims that he worked on Monday 22 April. Explain why he must be lying.

   Solution.
   Let us assume that Xavier's pay is B on certain day and we would like to know his pay, A, the day before (assuming that he worked the day before). If A has 3 as a factor, then $B = \frac{2}{3}A$ and B must be even. In this case $A = \frac{3}{2} \times B$. Otherwise, if 3 is not a factor of A, then $B = 2A + 1$ and so B is odd, and $A = \frac{1}{2}(B - 1)$.

   Consequently, if B is even then $A = \frac{3}{2} \times B$, and if B is odd, then $A = \frac{1}{2}(B - 1)$ (which must *not* have 3 as a factor). Now consider the days in reverse, beginning with Friday.

   (a) Friday – 60 is even so Thursday's pay is $\frac{3}{2} \times \$60 = \$90$.
   Thursday – 90 is even so Wednesday's pay is $\frac{3}{2} \times \$90 = \$135$.
   Wednesday – 135 is odd so Tuesday's pay is $\frac{1}{2}(\$135 - \$1) = \$67$.

   (b) Tuesday – \$67 is odd and $\frac{1}{2}(\$67 - \$1) = \$33$ has 3 as a factor so Xavier could *not* have worked on Monday.

4. On Monday, the cook used one teaspoon of salt in the soup, and it was not salty enough. On Tuesday, the cook used three times as much salt as on Monday, and the soup was still not salty enough. On Wednesday, it had three times as much salt as on Tuesday, and was too salty. The

amount of excess salt on Wednesday was equal to the average amount of salt lacking on Monday and on Tuesday. What was the correct amount of salt?

Solution.
Let x teaspoons be the correct amount of salt. Then

$9 - x$ teaspoons is the amount of excess salt on Wednesday,

$1 - x$ teaspoons is the amount of salt lacking on Monday,

$3 - x$ teaspoons is the amount of salt lacking on Tuesday.

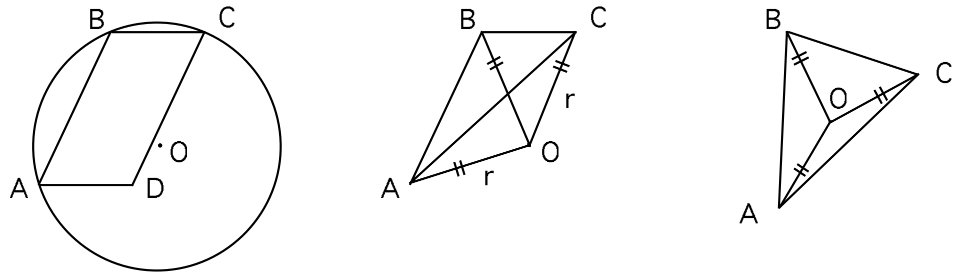
Thus

$$\begin{aligned} 9 - x &= \frac{(x-1)+(x-3)}{2}, \\ 18 - 2x &= 2x - 4, \\ x &= 5\tfrac{1}{2}. \end{aligned}$$

Alternatively, without using algebra, many students reasoned correctly that the average on Monday and Tuesday was 2 teaspoons, and the answer must be the average of this and Wednesday's 9 teaspoons; that is, $(2+9)/2 = 5\frac{1}{2}$.

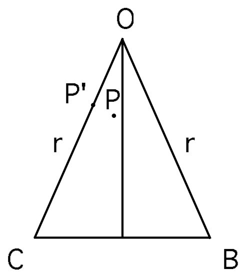
5. Let P be any point inside a parallelogram whose vertices are A, B, C and D, and let r be the radius of the circle through A, B and C. Explain why the distance from P to the nearest of the four vertices of the parallelogram cannot be greater than r.

   Solution.
   Notice that P lies either in triangle ABC or in triangle CDA. Assume firstly that P lies in ABC. Let O be the centre of the circle passing through A, B and C, and r its radius. O may lie inside the parallelogram, but also may lie outside as in the following figure (a):



(a) (b) (c)

There are two cases to consider: O lies outside triangle ABC as in (b), or inside, as in (c). In either case, any point P inside triangle ABC lies inside or on one of the triangles CAO or ABO or BCO; i.e, an isosceles triangle whose two equal sides have length r and whose base points are vertices of the original parallelogram. Without loss of generality we can consider the base CB as follows.



The angle CPO is clearly greater than $90°$, and so some point P' on CO creates an isosceles triangle CPP'. Thus the length of CP equals the length of CP' which is at most r. We conclude that the distance of P from the closest vertex of the parallelogram is at most r.

Finally, we need to treat the case that P lies in triangle CDA. But this triangle is equivalent (congruent) to ABC and so the same argument applies.

6. Show that the number of ways of distributing 101 identical coins into at most three piles is equal to the number of ways of distributing the same coins into any number of piles which have up to three coins each.

Solution.
Take any way of distributing the coins into at most three piles. This distribution gives us three numbers a, b and c where there are a coins in the largest pile, b in the second-largest and c in the smallest. Now take all the coins in the largest pile and make a piles of one coin each. Then take the b coins and add them one to each of b of the new piles. Finally, take the c coins and add one to each of c piles of two. This makes a new distribution of 101 coins into piles with at most three in each pile.

Note that each distribution of 101 coins into piles of at most three can be obtained through the above operations, from just one of the distributions of coins into at most three piles. So the number of distributions of the first type is equal to the number of the second type, as required.

To illustrate, the following figure shows two corresponding distributions of 16 coins, first into three piles, and then into piles with at most three coins in each ($a = 8$, $b = 7$, $c = 3$).