JUNIOR SOLUTIONS 2000

1. Six students, when asked the school's population, gave the answers 297, 305, 311, 315, 318 and 320. They were all wrong. The smallest error was 3 and the largest was 12. What was the correct answer?

   Solution.
   Since the largest error was 12 but the smallest and largest guesses differ by $320 - 297 = 23$, the true value must lie in between these guesses, and must be exactly 12 different from one of them. If it were $297 + 12 = 309$, the smallest difference would be $311 - 309 = 2$. So it must be $320 - 12 = 308$, which has the smallest error of 3 from 305 and 311.

2. Belinda wanted to buy her father a beard-trimmer. She offered five-twelfths of the money she had, but that was only 75% of the full price. So instead she offered two-thirds of her money, and received \$16 in change along with the trimmer. What was its price?

   Solution.
   From the first statement we find that Belinda's money is $\frac{12}{5} \times 75\%$, or $\frac{9}{5}$ times the cost of the trimmer. So $\frac{2}{3}$ of her money equals $\frac{6}{5}$ times the cost. She therefore recieves in change $\frac{1}{5}$ of the cost. That being \$16, the full cost must be \$80.

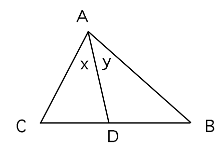
3. The embassy of Vacillatania has four flags: red, blue, yellow and green. Each day the ambassador flips a coin to decide which flag to put up the next day. If, one day, it is red, then the next day, it is blue or yellow (with equal chances). Also, if one day it is blue, then the next day it is blue or yellow (with equal chances). Similarly, if one day it is either yellow or green, then the next day it is green or red (with equal chances).

   What is the chance the flag is the same colour on Saturday as it was on Thursday?

   Solution.
   In half of the weeks when the flag is red on Thursday, it will be blue on Friday. If this happens, it cannot be red on Saturday. On the other half of the days when it is red on Thursday, it will be yellow on Friday, and in half of *these* occasions it will be red on Saturday, making one quarter of all occasions. So the chance it is red on Saturday, if it was red on Thursday, is one in four. The same calculation applies if it was green, blue or yellow on Thursday. So no matter what the colour on Thursday, the chance of being the same on Saturday is one in four.

4. A certain triangle can be sliced using one straight line cut to produce two smaller triangles with the same sizes of angles as the original triangle. Find those angles.

Solution.
Since the cut produces two smaller *triangles*, the cut must go through a vertex, which without loss of generality can be called A. Let the triangle be ABC, and let D denote the place where the cut intersects AB.



Let angle CAD be x, and angle DAB be y. Since the angles of triangle ABD are the same as ABC, and the angle at B is common to these two triangles, and since the angle CAB is larger than angle y, it must be that the angle ACB equals y. For the same reason, ABD equals x. But since angle CAB is $x + y$, the total of the three angles in triangle ABC is $180° = x + y + (x + y) = 2(x + y)$. So $x + y = 90°$. This means angle CAB is 90°. Since triangle ABD has the same angles as ABC, the angle ADB must also be 90°. Since $y = 90° - x$, it is now clear that the angle condition in the question is satisfied for any triangle ABC with one angle 90°, and the other two adding to 90°. In the figure above, AD should be drawn perpendicular to BC.

5. Tony went hiking to Tiger mountain. The path to the mountain was first flat, then steep uphill, then steep downhill to the lake. He returned along the same path, which is 12km each way in total. His speed was 5km/hour on the flat, 3km/hour uphill and 6km/hour downhill. If the whole trip took 5 hours and 12 minutes, how long was the flat section of the path?

   Solution.
   Let x km be the length of the flat section of the path. This is covered twice at 5km/hour (once on the forward journey and once on return), taking $2x/5$ hours. We are not told if the length of the steep section uphill is the same as downhill, but it does not matter: each part of the remaining $12-x$ km is covered once uphill (so uphill time is $(12 - x)/3$ hours) and once downhill ($(12 - x)/6$ hours).

$$\begin{aligned}
\text{Total time (in hours)} &= \frac{2x}{5} + \frac{12-x}{3} + \frac{12-x}{6} \\
&= \frac{6 \times 2x + 10 \times (12-x) + 5 \times (12-x)}{30} \\
&= \frac{12x + (120 - 10x) + (60 - 5x)}{30} \\
&= \frac{180 - 3x}{30} = 6 - \frac{x}{10}.
\end{aligned}$$

On the other hand, we are told that the total time is 5 hrs 12 minutes, or 5.2 hours. So

$$6 - \frac{x}{10} = 5.2, \quad \text{or} \quad 0.8 = \frac{x}{10}$$

and thus $x = 8$ is the length of the flat section (in km).

6. Mrs Brown has a number of daughters. Taking their ages (in whole years), she finds that the oldest is twice the average and the youngest is two-thirds of the average. If all the daughters have different ages and the oldest has not yet turned 33, how many daughters does she have?

   Solution.
   One way to approach this problem without a lot of guess-and-check work is to realise that the average age is quite a lot lower than the midpoint between the youngest and oldest. So the other daughters must tend to be closer to the youngest than the oldest, and this will force there to be a rather large number.

   If a is the average age then $\frac{2}{3}a$ is the age of the youngest, and $2a$ is the age of the oldest. But note that the oldest is exactly 3 timest the youngest, and is less than 33, so is at most 30. So $a \leq 15$.

   The youngest and oldest total $2\frac{2}{3}a$, and the next youngest daughter is at least $\frac{2}{3}a + 1$, the next at least $\frac{2}{3}a + 2$ and so on. Since $a \leq 15$, $1 \geq \frac{1}{15}a$, and these ages are at least $2\frac{2}{3}a + \frac{1}{15}a = \frac{11}{15}a$, $2\frac{2}{3}a + \frac{2}{15}a = \frac{12}{15}a$ and so on. So the total age, given the number of daughters, must be at least the following:

number of daughters	total age at least
3	$2\frac{2}{3}a + \frac{11}{15}a = 3\frac{6}{15}a$
4	$3\frac{6}{15}a + \frac{12}{15}a = 4\frac{3}{15}a$
5	$4\frac{3}{15}a + \frac{13}{15}a = 5\frac{1}{15}a$
6	$5\frac{1}{15}a + \frac{14}{15}a = 6a$
7	$6a + \frac{15}{15}a = 7a$
8	$7a + \frac{16}{15}a = 8a + \frac{1}{16}$

   and so on. But with n daughters, the total age must be exactly na (that's by definition of "average"), which is impossible by this table if n is less than 6, or if n is 8 or more since each extra daughter adds on more than a to the total.

   Finally, the table helps us find the ages. Since we can only *just* achieve the correct average for $n = 6$ and 7, the maximum value $a = 15$ must be achieved. If there are 6 daughters the ages are 10, 11, 12, 13, 14, 30, and if 7 daughters, 10, 11, 12, 13, 14, 15, 30. So the final answer is: she has 6 or 7 daughters, both are possible given the information.

7. From the year 0 onwards, priests in the kingdom of Sequencia have performed a ceremony involving a row of one thousand tiles which are black on one side and white on the other. Once a year, if the left-most tile in the row is white, all the tiles are flipped over, whilst if it is black, that tile alone is flipped over and transferred to the other end of the row. For example, if the row had only five tiles, then WBBWB (W = white, B = black) would the next year become BWWBW, and the year after that WWBWW.

   Show that the pattern on the thousand tiles today has already occurred in one of the 2000 earlier years (regardless of the starting pattern).

   Solution.
   The key to answering this is to ignore the colours on the tiles! Write 0 on the face of each tile that is showing at year 0, and 1 on the opposite faces. Then the starting pattern is a sequence of 1000 0's. Each year, one of two operations occurs, which affect the sequence as follows: if all the tiles are flipped over, then every 0 changes to a 1 and vice versa. If the left-most tile is flipped over and transferred to the right, then the left-most member of the sequence is changed (0 becomes 1, and 1 becomes 0) and put at the right-most end. Thus for instance if the row of tiles had length 5, 00000 can become 11111 or 00001, and 11111 can become 00000 or 11110, and 00001 can become 11110 or 00011. To make it clearer what is happening, we can show the position in the sequence of the original left-most tile by writing * to its left. So the original sequence is *00000 and the others shown above are *11111, 0000*1, 1111*0 and 000*11.

   It is clear that even with a thousand tiles, every sequence obtained has all numbers to the left of the * equal, all numbers to the right of the * equal, and those on opposite sides of the * different (as this property is maintained when carrying out either of the two operations). So the position of the first tile, and the number showing on its face, determine the whole sequence. Since there are a thousand positions for the first tile and then two choices of which number it shows, that makes 2000 possible sequences. So by the year 2000, (which would make 2001 sequences), some sequence must have repeated, and hence the black-white pattern has repeated as well (since the sequence, and the position of the *, determine the black-white pattern). From the first year that a pattern occurs the same as in an earlier year (say x years beforehand), the pattern in evey subsequent year is the same as x years beforehand. So the pattern in year 2000 is the same as at some time in the past 2000 years.