

**UNIVERSITY OF MELBOURNE SCHOOL
MATHEMATICS COMPETITION, 2002:
INTERMEDIATE DIVISION—PROBLEMS AND
SOLUTIONS**

1. A pizza of diameter 8 cm is cut into 3 equal slices, one of diameter 10cm is cut into 4 equal slices, one of diameter 12cm is cut into 6 equal slices and another of diameter 14cm is cut into 8 equal slices. From which pizza should you choose a slice, assuming you are very hungry?

Solution: Assuming that the pizzas are flat, of equal thickness and that you are very hungry, you would choose the slice of pizza which had the largest area.

- 8 cm pizza

$$\text{Area}(\text{pizza}) = \pi r^2 = \pi 4^2 \text{cm}^2 = 16\pi \text{cm}^2$$

$$\text{Area}(\text{slice}) = 16\pi \div 3 \text{ cm}^2 = \frac{16}{3}\pi \text{cm}^2$$

- 10 cm pizza

$$\text{Area}(\text{pizza}) = \pi r^2 = \pi 5^2 \text{cm}^2 = 25\pi \text{cm}^2$$

$$\text{Area}(\text{slice}) = 25\pi \div 4 \text{ cm}^2 = \frac{25}{4}\pi \text{cm}^2$$

- 12 cm pizza

$$\text{Area}(\text{pizza}) = \pi r^2 = \pi 6^2 \text{cm}^2 = 36\pi \text{cm}^2$$

$$\text{Area}(\text{slice}) = 36\pi \div 6 \text{ cm}^2 = 6\pi \text{cm}^2$$

- 14 cm pizza

$$\text{Area}(\text{pizza}) = \pi r^2 = \pi 7^2 \text{cm}^2 = 49\pi \text{cm}^2$$

$$\text{Area}(\text{slice}) = 49\pi \div 8 \text{ cm}^2 = \frac{49}{8}\pi \text{cm}^2$$

However, since $\frac{25}{4} > \frac{49}{8} > 6 > \frac{16}{3}$, the slice of pizza with the largest area is the one cut from the pizza of diameter 10 cm.

2. A shop offered triple the GST in savings. A salesgirl calculated the selling price by first reducing the original price by 30% and then adding the 10% GST based on the reduced price. A customer protested, saying that the store should first add the 10% GST and then reduce that total by 30%. They agreed on a compromise: the salesgirl reduced the original price by 20%. How do the three ways compare with one another from the customer's point of view?

- (a) The salesgirl's way is the best.
- (b) The customer's way is the best.
- (c) The compromise is the best.
- (d) All three ways are the same.
- (e) The compromise is the worst while the other two are the same.

Solution: Let x denote the original selling price of the item.

- *Salesgirl:* Reducing the price of the item by 30% brings the price to 70% of its original value: $0.7x$. Adding 10% GST brings the price to 110% of its reduced value: $1.1 \times 0.7x = 0.77x$.
- *Customer:* Adding 10% GST brings the price to 110% of its original value: $1.1x$. Reducing the price of the item by 30% brings the price to 70% of its value after tax: $0.7 \times 1.1x = 0.77x$.
- *Compromise:* Reducing the original price by 20% brings the price to 80% of its original value: $0.8x$.

Hence, from the customer's point of view, the compromise is the worst while the other two are the same.

3. (a) Determine the smallest positive integer n such that $n^3 + 2n^2$ is the square of an odd integer.
- (b) Determine the smallest such integer n greater than 2002.

Solution: If n is even, then $n^3 + 2n^2 = n^2(n + 2)$ will also be even, and hence, cannot be the square of an odd integer. Thus, n must be odd. By Euclid's Algorithm:

$$\begin{aligned}
 & \gcd(n^2, n + 2) \\
 &= \gcd(n^2, n^2 - (n - 2)(n + 2)) \\
 &= \gcd(n^2, 4) \\
 &= 1
 \end{aligned}$$

since we have already determined that n is odd. So n^2 and $n + 2$ are relatively prime integers whose product is the square of an odd integer. Therefore, n^2 and $n + 2$ must both also be squares of odd integers.

Since n is positive, $n + 2 \geq 3$ and the smallest candidate for n is when $n + 2 = 3^2 = 9$. This gives $n = 7$ as the smallest possible integer

satisfying the given conditions, and it is not hard to verify that $n = 7$ is indeed the solution:

$$7^3 + 2 \times 7^2 = 7^2 \times (7 + 2) = 7^2 \times 3^2 = 21^2$$

Note that $44^2 = 1936 < 2002 < 2025 = 45^2$, so the smallest candidate for $n > 2002$ is when $n + 2 = 45^2 = 2025$. This gives $n = 2023$ as the smallest possible integer satisfying the given conditions, and it is not hard to verify that $n = 2023$ is indeed the solution:

$$2023^3 + 2 \times 2023^2 = 2023^2 \times (2023 + 2) = 2023^2 \times 45^2 = (2023 \times 45)^2$$

4. A right angled triangle has a circumscribing circle of radius R and an inscribed circle of radius r . If the perpendicular sides of the triangle are of length 16cm and 30 cm, find R and r .

Solution: A chord in a circle subtends an angle of 90° if and only if it is a diameter. Therefore, the hypotenuse of the triangle must also be a diameter and have length $2R$. So by Pythagoras' Theorem:

$$\begin{aligned} 16^2 + 30^2 &= (2R)^2 \\ 256 + 900 &= (2R)^2 \\ 1156 &= (2R)^2 \\ 2R &= 34 \\ R &= 17 \end{aligned}$$

Consider an arbitrary triangle ABC with an inscribed circle of radius r . Let the centre of the inscribed circle be I and let the perpendiculars from I meet BC , CA , AB at D , E , F , respectively. Then we can deduce the formula:

$$\begin{aligned} \text{Area}(\triangle ABC) &= \text{Area}(\triangle BIC) + \text{Area}(\triangle CIA) + \text{Area}(\triangle AIB) \\ &= \frac{1}{2}BC \cdot DI + \frac{1}{2}CA \cdot EI + \frac{1}{2}AB \cdot FI \\ &= \frac{1}{2}BC \cdot r + \frac{1}{2}CA \cdot r + \frac{1}{2}AB \cdot r \\ &= \frac{1}{2}(AB + BC + CA) \cdot r \end{aligned}$$

Using this formula for the given triangle, we have:

$$\begin{aligned}\text{Area}(\triangle ABC) &= \frac{1}{2}(AB + BC + CA).r \\ \Rightarrow \frac{1}{2} \times 16 \times 30 &= \frac{1}{2}(16 + 30 + 34).r \\ \Rightarrow 40r &= 240 \\ \Rightarrow r &= 6\end{aligned}$$

So R is 17 cm and r is 6 cm.

5. Which of the following numbers cannot be expressed as the difference between the squares of two integers. (For example $20 = 36 - 16 = 6^2 - 4^2$ can be so expressed.)

- (a) 314159265
- (b) 314159266
- (c) 314159267
- (d) 314159268
- (e) 314159269

Solution: We first note that any odd number can be written in the form $2n + 1$ and can be expressed as the difference of perfect squares as follows:

$$(n + 1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1.$$

Also, any number divisible by 4 we can write in the form $4n$ and can be expressed as the difference of perfect squares as follows:

$$(n + 1)^2 - (n - 1)^2 = (n^2 + 2n + 1) - (n^2 - 2n + 1) = 4n.$$

Thus, all integers congruent to 0, 1 or 3 (modulo 4) can be expressed as the difference of perfect squares. We will now show that all integers congruent to 2 (modulo 4) cannot be expressed in such a way.

Note that $0^2 \equiv 0, 1^2 \equiv 1, 2^2 \equiv 0, 3^2 \equiv 1 \pmod{4}$. So perfect squares can only be congruent to 0 or 1 (modulo 4). Thus, differences of perfect squares can only be congruent to $0 - 0 \equiv 0, 0 - 1 \equiv 3, 1 - 0 \equiv 1, 1 - 1 \equiv 0 \pmod{4}$. So a difference of perfect squares can never be congruent to 2 (modulo 4). Thus, all integers congruent to 2 (modulo 4) cannot be expressed as the difference of perfect squares.

So we have now proven that the only numbers that can be expressed as the difference of perfect squares are those congruent to 0, 1 or 3 (modulo 4).

$$314159265 \equiv 3141592 \times 100 + 65 \equiv 65 \equiv 1 \pmod{4}$$

$$314159266 \equiv 3141592 \times 100 + 66 \equiv 66 \equiv 2 \pmod{4}$$

$$314159267 \equiv 3141592 \times 100 + 67 \equiv 67 \equiv 3 \pmod{4}$$

$$314159268 \equiv 3141592 \times 100 + 68 \equiv 68 \equiv 0 \pmod{4}$$

$$314159269 \equiv 3141592 \times 100 + 69 \equiv 69 \equiv 1 \pmod{4}$$

So, of the five numbers given, only 314159266 cannot be expressed as the difference of perfect squares whereas the others can be.

6. A and B are two points on the diameter MN of a semicircle. The points C , D , E and F on the semicircle are positioned such that $\angle CAM = \angle EAN = \angle DBM = \angle FBN$. Prove that $CE = DF$.

Solution: Let Γ denote the circle with diameter MN and let C' , D' , E' , F' be the reflections of C , D , E , F in the line MN . Clearly, C' , D' , E' , F' lie on Γ by symmetry.

First, we note that $CD = C'D'$ by symmetry. Also, since $\angle EAN = \angle FBN$, EA is parallel to FB . In particular, EC' and FD' are parallel chords of Γ . So by symmetry, $C'D' = EF$. Therefore, we deduce that $CD = EF$.

Thus, the arc CD is equal in length to the arc EF . So we can obtain the arc EF by rotating the arc CD by some angle θ about the centre of Γ . Hence, the chords CE and DF both subtend the angle θ at the centre of Γ and are the same length.

7. Suppose that $p > 2$ is a prime number, and a and b are positive integers such that p^5 divides $a^2 + b^2$ and p^5 also divides $a(a + b)^2$. Is it true that p^5 necessarily divides $a(a + b)$? Either prove it is true, or provide a counterexample.

Solution: Consider $a = 25$, $b = 50$, $p = 5$. Then:

$$\begin{aligned} p^5 &= 5^5 = 3125 \\ a^2 + b^2 &= 25^2 + 50^2 = 3125 \\ \Rightarrow p^5 &| a^2 + b^2 \end{aligned}$$

$$\begin{aligned}
p^5 &= 5^5 = 3125 \\
a(a+b)^2 &= 25(25+50)^2 = 25 \times 75^2 = 140625 = 45 \times 3125 \\
\Rightarrow p^5 &| a(a+b)^2
\end{aligned}$$

$$\begin{aligned}
p^5 &= 5^5 = 3125 \\
a(a+b) &= 25(25+50) = 25 \times 75 = 1875 \\
\Rightarrow p^5 &\nmid a(a+b)
\end{aligned}$$

This counterexample shows that it is not necessarily true that p^5 divides $a(a+b)$ under the given conditions.