

(1) If Samantha had arrived home two hours later than she did, it would have been one-third as long until midnight as it would have been had she arrived home only one hour later. At what time did Samantha arrive home?

Solution: Suppose that Samantha arrived home x hours before midnight. Then $x - 2 = \frac{1}{3}(x - 1)$ so that $x = 2.5$. Therefore Samantha arrived home at 9:30pm.

(2) Jing and Tian play a game in which they take it in turns to choose three integers between 2 and 10 (inclusive). Jing chooses the first number. As they progress through the game they add their own numbers together and their corresponding cumulative totals are always prime numbers. At the end of the game it is found that the six numbers chosen are all different.

Also, Jing had a greater cumulative total than Tian after each had chosen one number, and also after each had chosen their three numbers. After each had chosen two numbers, Tian's cumulative total exceeded that of Jing's. Find the numbers chosen by each person.

(A prime number is one whose only factors are 1 and the number itself).

Solution: The prime numbers between 2 and 10 are 2,3,5 and 7. If Jing's first choice is 3, then Tian's first choice must be 2. It is then easy to show that subsequent choices cannot satisfy the conditions of the game.

A systematic analysis shows that

- Jing's choices (in order) are 5,6,8, and
- Tian's choices (in order) are 3,10, 4.

(3) Solve the following “cross-number” puzzle in which each entry is one of the integers 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9. No answer can begin with “0”.

1	2	3
4		
5		

The clues are:

<u>Across</u>	<u>Down</u>
1. A prime number.	1. A value of 5^a .
4. Not a prime number.	2. A value of 2^b .
5. A number which is a square	3. A value of 3^c . (a, b, c are positive integers)

Solution: 1Down must be 125 or 625 and so 5Across must be 529 or 576. 3Down must be 243 or 729, so 729 is suitable. Continuing in this manner leads to the following solution:

1	5	7
2	1	2
5	2	9

(4) Richard has a selection of identical blue marbles, identical green marbles and identical red marbles. Richard finds that 2 green marbles weigh the same as 2 red marbles together with 1 blue marble, while 1 green marble together with 1 red marble weigh the same as 2 blue marbles. What combinations of red and green marbles weigh the same as 5 blue marbles?

Solution: Let b , g and r represent, in appropriate units, the weights of blue, green and red marbles respectively. Then

$$2r + b = 2g \quad \text{and} \quad (1)$$

$$g + r = 2b \quad (2)$$

Substituting $r = 2b - g$ from (2) into (1) gives $4g = 5b$. It also follows from (1) and (2) that $3g = 5r$, whence

$$5b = 4g = g + 3g = g + 5r.$$

Hence 4 green marbles, or 1 green marble together with 5 red marbles, weigh the same as 5 blue marbles.

(5) Find all three-digit positive integers that are equal to 16 times the sum of their digits.

Solution: Let the three digit number be “ xyz ”, which can be written $100x + 10y + z$. Then

$$100x + 10y + z = 16(x + y + z),$$

which can be simplified to

$$2(14x - y) = 5z.$$

Hence we can write $z = 2t$ and $14x - y = 5t$, for some positive integer t . However since $0 \leq z \leq 9$, the possible values of z are 0, 2, 4, 6 or 8.

- If $z = 0$ then $y = 14x$ which does not provide any solutions.
- If $z = 2$ then $t = 1$ and $y = 14x - 5$. The only solution is therefore $x = 1, y = 9$, giving 192.

Continuing in this way generates two other solutions 144 and 288.

(6) Find all positive integers n such that $\frac{(n+1)^2}{n+23}$ is also a positive integer.

Solution: Write

$$\frac{(n+1)^2}{n+23} = n - 21 + \frac{484}{n+23},$$

and observe that $484 = 2 \times 242 = 4 \times 121 = 11 \times 44$. It follows that the only solutions are 21, 98, 219 and 461.

(7) An “*interesting*” rectangular box is one whose sides have integer lengths, and which has the property that its surface area is numerically equal to its volume. Find the dimensions of all “*interesting*” rectangular boxes.

Solution: Let the dimensions of the box be, in appropriate units, x, y, z , with $x \geq y \geq z$. Then

$$2(xy + yz + zx) = xyz. \quad (1)$$

Therefore $xyz \leq 6xy$ and so $z \leq 6$.

Also

$$xyz - 2xy = xy(z - 2) = 2yz + 2zx \geq 0 \quad (2)$$

Hence $z \geq 3$ and so $z = 3, 4, 5$ or 6 .

From (2):

$$xy = \frac{2z(x+y)}{z-2} \leq \frac{4zx}{z-2}.$$

and so $y \leq \frac{4z}{z-2}$.

From (1):

$$x = \frac{2yz}{yz - 2y - 2z} > 0$$

and so $yz - 2y - 2z > 0$ and $y > \frac{2z}{z-2}$.

Combining these inequalities gives

$$\frac{2z}{z-2} < y \leq \frac{4z}{z-2} \quad (3)$$

Using these inequalities we can work through the possible values of z starting with $z = 3$. The choices for y are then 7, 8, 9, 10, 11 or 12. If $y = 11$ then $x = 66/5$ which is not an integer. All other choices for y are acceptable giving solutions (42, 7, 3), (24, 8, 3), (18, 9, 3), (15, 10, 3) and (12, 12, 3).

Similar analyses for $z = 4, 5$ and 6 give the solutions (20, 5, 4), (12, 6, 4), (8, 8, 4), (10, 5, 5) and (6, 6, 6).