2006 Junior Competitions (Years 7 and 8); Solutions

1. Sudoku. In the top-right square we must add a 4 and a 2. Since the second row already has a 4, this can only be done as

		1	4
4		3	2
2			

Next we can complete all the ones as

		1	4
4	1	3	2
1			
2			1

After a few more steps we find

3	2	1	4
4	1	3	2
1	4	2	3
2	3	4	1

2. Number crunch. Let the five distinct numbers be $x < y < z < u < v$. Then $x + y = 8$, $x + z = 11$ and also $u + v = 21$, $z + v = 19$. Thus $y = 8 - x$, $z = 11 - x$, $v = 19 - z = 19 - (11 - x) = 8 + x$ and $u = 21 - v = 21 - (8 + x) = 13 - x$. The 5 numbers may therefore be written as $x < 8 - x < 11 - x < 13 - x < 8 + x$. Since $13 - x < 8 + x$ we must have that $x \geq 3$. Since $x < 8 - x$ we must have that $x \leq 3$. Therefore, $x = 3$ and the numbers are $3, 5, 8, 10, 11$.

3. Tennis woes. From (a) we have that a court is rectangular. Let the sides (in metres) of this rectangle be x and y, with $x \geq y$. After folding the rectangle we get a new rectangle of dimensions $x/2$ and y. According to (b) the two rectangles are similar. Hence

$$\frac{y}{x} = \frac{x/2}{y}.$$

Multiplying both sides by x and y we get $y^2 = x^2/2$ so that $y = x/\sqrt{2}$. The area (in square metres) of the court is $x \times y$ and according to (c) this must be 200. We thus have two equations:

$$x \times y = 200 \quad \text{and} \quad y = x/\sqrt{2}.$$

Therefore

$$200 = x \times y = x \times x/\sqrt{2} = x^2/\sqrt{2}.$$

In other words, $x^2 = 200\sqrt{2}$ so that

$$x = \sqrt{200\sqrt{2}} = 10\sqrt{2\sqrt{2}} = 10 \cdot 2^{3/4}$$

and

$$y = x/\sqrt{2} = 10\sqrt{2\sqrt{2}}/\sqrt{2} = 10\sqrt{\sqrt{2}} = 10 \cdot 2^{1/4}$$

4. Squares and their divisors. If d is a divisor of n then so is the number n/d. For example, since 4 is a divisor of 20, so is the number $20/4 = 5$. Hence all divisors come in pairs and each number has an even number of divisors. For example, the 3 pairs of divisors of 20 are $(1, 20), (2, 10), (4, 5)$, so that 20 has $2 \times 3 = 6$ divisors.

If a number n is square then the above reasoning is incorrect since in this case we get the pair $(\sqrt{n}, \sqrt{n})$ which should be counted as a single divisor only. For example, the divisors of 36 are the pairs $(1, 36), (2, 18), (3, 12), (4, 9)$ plus the solitary divisor $\sqrt{36} = 6$, resulting in $2 \times 4 + 1 = 9$ divisors. Hence squares have an odd number of divisors and all other numbers have an even number of divisors.

5. Yin and Yang. Since we are computing ratios of areas of semicircles we only need to use that Area of a semicircle $= Cd^2$ with d the diameter of the semicircle.

The 2 big semicircles (making up the big circle) both have area $B = C(x+y)^2$. The two small semicircles have area $S_1 = Cx^2$ and $S_2 = Cy^2$.

The area of the left side is

$$\begin{aligned} B - S_1 + S_2 &= C[(x+y)^2 - x^2 + y^2] = C[(x^2 + 2xy + y^2) - x^2 + y^2] \\ &= C[2y^2 + 2xy] = 2Cy(x+y). \end{aligned}$$

The area of the right side is

$$\begin{aligned} B + S_1 - S_2 &= C[(x+y)^2 + x^2 - y^2] = C[(x^2 + 2xy + y^2) + x^2 - y^2] \\ &= C[2x^2 + 2xy] = 2Cx(x+y). \end{aligned}$$

The ratio between left and right is therefore

$$\frac{2Cy(x+y)}{2Cx(x+y)} = \frac{y}{x}.$$

6. Squares. Let the sides of the biggest square be 1. Then the total area of the big square is $1 \times 1 = 1$. The area b_1 of the biggest black square is a quarter of this, i.e., $b_1 = 1/4$. The area b_2 of the second-biggest black square is a quarter of b_1. Hence $b_2 = 1/16 = 1/4^2$. The area b_3 of the third-biggest black square is a quarter of b_2. Hence $b_3 = 1/64 = 1/4^3$, and so on. Hence the total area in black is

$$\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots.$$

On the other hand, on top and to its right, each black square has equal-sized white squares. Together these 'hooks' build up the big square. Since a third of the area of each hook is black it follows that a third of the total area of the big square is black. Hence

$$\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots = \frac{1}{3}.$$

7. Footy colours. Let (x, y, z) be the number of Carlton, Collingwood and Cats jumpers at any given time during the game. For example, at the start of the game $(x, y, z) = (6, 8, 10)$. Now (x, y, z) can change in 3 possible ways

$$\begin{aligned} (x, y, z) &\to (x+2, y-1, z-1) \\ (x, y, z) &\to (x-1, y+2, z-1) \\ (x, y, z) &\to (x-1, y-1, z+2). \end{aligned}$$

Note that the difference between the number of Collingwood and Carlton jumpers always changes by a multiple of 3. For example, beginning with $(6, 8, 10)$ (so that the difference is $8 - 6 = 2$) we can change to $(8, 7, 9)$ (so that the difference is $7 - 8 = -1 = 2 - \mathbf{3}$) or $(5, 10, 9)$ (so that the difference is $10 - 5 = 5 = 2 + \mathbf{3}$) or $(5, 7, 12)$ (so that the difference is $7 - 5 = 2 = 2 + \mathbf{0}$). The same applies to the difference between Cats and Collingwood and between Cats and Carlton jumpers.

If we try to end up with only Carlton jumpers then the final numbers would have to be $(24, 0, 0)$ with differences between the number of jumpers all a multiple of 3. Since these differences change by multiples of 3 we should also start the game with differences being multiples of 3. But that is clearly not the case since the initial differences are 2, 2 and 4. Hence we cannot finish with only Carlton jumpers. The same reasoning applies to finishing with only Collingwood or Cats jumpers.

The conclusion is that St Kilda will win the 2006 flag.