1. The four-digit number 3025 can be split into two, so that its first two digits are 30, and its second two are 25. If we add these two numbers, giving 55, and square it, we get 3025, the original number again. Can you find another four digit number with the same property? (After performing the split, the two numbers must be greater than or equal to 10. For example, 9801 = (98 + 1)^2 is not allowed.)

Solution: The only solution to this question is 2025 = (20 + 25)^2. One rather tedious way to obtain this answer is to test all four-digit numbers. This method becomes slightly more tractable after one realises that it is only necessary to test those four-digit numbers which are also perfect squares. A more refined method is as follows.

Suppose that we split our four-digit number $X$ into the two-digit number $a$ and the two-digit number $b$ so that $X = 100a + b$. We are looking for two-digit numbers $a$ and $b$ which satisfy the equation

$$(a + b)^2 = 100a + b.$$ 

Rearranging the equation, we obtain

$$a^2 + (2b - 100)a + (b^2 - b) = 0,$$

which is clearly a quadratic in the variable $a$.

We are given that $(a, b) = (30, 25)$ is a solution to this equation, so that 30 must be a root of the quadratic equation

$$a^2 - 50a + 600.$$ 

However, recall that a quadratic equation with leading coefficient 1 has two roots which multiply to give the constant term. Therefore, 20 is also a root of this quadratic equation which yields the solution $(a, b) = (20, 25)$ to the original equation. So we obtain 2025 as a number satisfying the given conditions.

2. Four people, Andrew, Barry, Christine and Dianna must cross a bridge in 17 minutes. All four begin on the same side of the bridge. It is midnight and very dark. There is one torch, which must be carried with each crossing. A maximum of two people can cross at any one time. The torch must be carried at all times (e.g., no throwing). Each person walks at different speeds. Andrew takes 5 minutes to cross, Barry takes 2 minutes to cross, Christine takes 10 minutes to cross, and Dianna takes 1 minute to cross. How can they cross the bridge in 17 minutes? (For example, if Christine and Dianna cross together, with the torch, and Dianna returns, then crosses with Andrew, and returns again to finally cross with Barry, the time taken is $10 + 1 + 5 + 1 + 2 = 19$ minutes.)
Solution: The four people can cross the bridge in 17 minutes by using the following method.

- Barry and Dianna cross — this takes 2 minutes.
- Barry returns — this takes 2 minutes.
- Andrew and Christine cross — this takes 10 minutes.
- Dianna returns — this takes 1 minute.
- Barry and Dianna cross — this takes 2 minutes.

Therefore, the total amount of time taken is $2 + 2 + 10 + 1 + 2 = 17$ minutes. Furthermore, it turns out that this is the minimum amount of time in which the four people can cross the bridge.

3. Consider a $10 \times 10$ chessboard. Can it be covered exactly by twenty-five tiles, all in the shape of a $4 \times 1$ rectangle?

Solution 1: Let a square of the chessboard be called groovy if it is in an odd numbered row and an odd numbered column. It is easy to check that every $4 \times 1$ rectangle on the chessboard covers either 0 or 2 groovy squares. Hence, it is only possible to cover an even number of groovy squares, whereas there are 25 groovy squares on the chessboard. It follows that the required tiling is impossible.

Solution 2: Suppose that we label each square in the chessboard with 0, 1, 2 or 3 in the following way. A square in the $m$th row and $n$th column is assigned the remainder when $m - n$ is divided by 4. It should be easy to see that every time we place a $4 \times 1$ rectangle on the chessboard, it covers exactly one square of each label. Therefore, if it were possible to exactly cover the chessboard with $4 \times 1$ rectangles, then there would have to be the same number of squares with each label. However, a simple count reveals that there are 26 squares labelled 0, 25 squares labelled 1, 24 squares labelled 2 and 25 squares labelled 3. Therefore, the required tiling is impossible.

Solution 3: Number the first row of the chessboard 1212121212, the next row 3434343434, and repeat this sequence 5 times. Each $4 \times 1$ rectangle must cover each number exactly two or zero times. As in the first solution, it is therefore possible to cover any integer only an even number of times. But each integer occurs 25 times, so the required tiling is impossible.

4. The circle $x^2 + y^2 = 25$ intersects the $x$-axis at points $A$ and $B$. The line $x = 11$ intersects the $x$-axis at the point $C$. Point $P$ moves along the line $x = 11$ above the $x$-axis, and $AP$ intersects the circle at $Q$.

(a) Determine the coordinates of $P$ when triangle $AQB$ has maximum area. Justify your answer.

(b) Determine the coordinates of $P$ when $Q$ is the midpoint of $AP$. Justify your answer.
(c) Determine the coordinates of $P$ when the area of $AQB$ is $\frac{1}{4}$ of the area of triangle $APC$. Justify your answer.

Solution: We begin by noting that $A = (-5, 0), B = (5, 0)$ and $C = (11, 0)$.

(a) The triangle $AQB$ always has the diameter $AB$ as its base. Therefore, its area is maximised when its height is maximised. Since $Q$ is constrained to the upper part of the circle, the height of triangle $AQB$ is maximised when $Q = (0, 5)$. In this case, the line $AQ$ has the equation $y = x + 5$ and it intersects the line $x = 11$ at $P = (11, 16)$.

(b) Since $A$, $Q$ and $P$ are collinear, the point $Q$ will be the midpoint of $AP$ precisely when the $x$-value of $Q$ is halfway between the $x$-values of $A$ and $P$. Since the $x$-value of $A$ is $-5$ and the $x$-value of $P$ is $11$, the $x$-value of $Q$ will be $\frac{-5 + 11}{2} = 3$. Furthermore, we require $Q$ to be on the circle, and by Pythagoras’ Theorem, we deduce that $Q = (3, 4)$. Therefore the line $AQ$ has the equation $y = \frac{1}{2}x + \frac{5}{2}$ and it intersects the line $x = 11$ at $P = (11, 8)$.

(c) Note that triangle $AQB$ and triangle $ACP$ are similar since they are both right-angled and share a common angle at $A$. Since we require triangle $ACP$ to have 4 times the area, its side lengths must be precisely twice as long as the corresponding sides in triangle $AQB$. In particular, this means that $AP = 2AB = 20$. By using Pythagoras’ Theorem in triangle $ACP$, we obtain $PC^2 = AP^2 - AC^2 = 20^2 - 16^2 = 144$, so $PC = 12$. Therefore, $P = (11, 12)$.

5. Recall that a trapezoid is a quadrilateral with one pair of sides parallel, and the other pair of sides not parallel. Let the lengths of the parallel sides be $a$ and $b$. A line segment is drawn, parallel to these two sides, that divides the trapezoid into two trapezoids of equal area. Determine the length of this line.

Solution: Let us assume without loss of generality that $a < b$ and let $x$ be the length of the desired line segment. In the diagram below, we have drawn the trapezoid and constructed a parallelogram with the same height and with base $x$. Let the upper and lower trapezoids have heights $m$ and $n$, respectively.
The area of the upper trapezoid is \( \frac{(a+x)m}{2} \) while the area of the lower trapezoid is \( \frac{(b+x)n}{2} \). Since these two areas are equal, we have

\[
\frac{(a+x)m}{2} = \frac{(b+x)n}{2} \Rightarrow \frac{m}{n} = \frac{b + x}{a + x}.
\]

Now note that the two small triangles in the diagram above are similar. The upper triangle has base \( x - a \) and height \( m \) while the lower triangle has base \( b - x \) and base \( n \). Therefore, we obtain the equal ratios

\[
\frac{m}{n} = \frac{x - a}{b - x}.
\]

We can equate these two expressions for \( \frac{m}{n} \) to obtain the following equation.

\[
x = \sqrt{\frac{a^2 + b^2}{2}}
\]

6. Let \( a, b, c \) be real numbers greater than 1, and let

\[
S = \log_a bc + \log_b ca + \log_c ab,
\]

where \( \log_b x \) denotes the base \( b \) logarithm of \( x \). Find the smallest possible value of \( S \).

**Solution:** To tackle this question, one must be familiar with the following two logarithm laws.

- \( \log_X Y = \frac{\log Y}{\log X} \), where the base of the logarithms on the right hand side are equal to any real number greater than 1
- \( \log_X Y Z = \log_X Y + \log_X Z \)

Using these two laws, we can write

\[
S = \log_a bc + \log_b ca + \log_c ab \\
= \log bc + \log ca + \log ab \\
= \log b + \log c + \log a + \log b + \log c \\
= (\log b + \log c) + (\log b + \log c) + (\log a + \log b) \\
= (x + \frac{1}{x}) + (y + \frac{1}{y}) + (z + \frac{1}{z})
\]
where we have used the substitution $x = \frac{\log b}{\log a}$, $y = \frac{\log c}{\log b}$, $z = \frac{\log a}{\log c}$.

Since $a, b, c > 1$, we have $x, y, z > 0$. However, for $x > 0$, we have the inequality $x + \frac{1}{x} \geq 2$ which can be proved as follows.

$$\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 \geq 0$$

$$\Rightarrow x - 2 + \frac{1}{x} \geq 0$$

$$\Rightarrow x + \frac{1}{x} \geq 2$$

Of course, a similar inequality holds for $y$ and $z$, so it is clear that

$$S = \left(x + \frac{1}{x}\right) + \left(y + \frac{1}{y}\right) + \left(z + \frac{1}{z}\right) \geq 6.$$}

Furthermore, if $a = b = c$, then $x = y = z = 1$ which yields the value $S = 6$. Therefore, the smallest possible value of $S$ is 6.

7. Find the smallest integer $n > 4$ such that there exists a set of $n$ people satisfying the following two conditions:

(a) any pair of acquainted people have no common acquaintance, and
(b) any pair of unacquainted people have exactly two common acquaintances.

(Note that acquaintance is symmetrical. In other words, if $A$ is acquainted with $B$, then $B$ is acquainted with $A$.)

Solution: Pick a person $A$, called the primary person. Suppose that $A$ is acquainted with $k$ people, called the secondary people. Suppose that there are $m$ people remaining, called the tertiary people. Therefore, there are $1 + k + m$ people in total.

First, note that any two secondary people cannot be acquainted. Therefore, they must have exactly two common acquaintances. One of them is $A$, so the other must be a tertiary person. Also, each tertiary person is not acquainted with $A$, so they must be acquainted with precisely two secondary people. So for every pair of secondary people, there is precisely one tertiary person acquainted with them. It follows that $m = \binom{k}{2} = \frac{k(k-1)}{2}$. Therefore, the total number of people is $n = 1 + k + \frac{k(k-1)}{2} = k^2 + k + 2$.

Since the primary person was chosen at random, the same argument applies to any other person. It follows that if one person is acquainted with $k$ people, then everyone else is acquainted with $k$ people.

Since $n > 4$, we need only consider $k \geq 3$. For $k = 3$, we have $n = 7$. This is not possible, since the total number of pairs of people who are acquainted is $\frac{n(n-1)}{2}$, which is not an integer in this case.
For $k = 4$, we have $n = 11$. There are six tertiary people $C_1, C_2, C_3, C_4, C_5, C_6$, each of whom is acquainted with precisely two secondary people $B_1, B_2, B_3, B_4$. Without loss of generality, we may assume that $C_1$ is acquainted with $B_1$ and $B_2$, $C_2$ is acquainted with $B_1$ and $B_3$, $C_3$ is acquainted with $B_1$ and $B_4$, $C_4$ is acquainted with $B_2$ and $B_3$, $C_5$ is acquainted with $B_2$ and $B_4$, and $C_6$ is acquainted with $B_3$ and $B_4$.

Since everyone is acquainted with exactly $k = 4$ people, $C_1$ must be acquainted with two other people from the group $C_1, C_2, C_3, C_4, C_5, C_6$. However, $C_1$ has a common acquaintance with $C_2, C_3, C_4, C_5$ so the only other person $C_1$ can be acquainted with is $C_6$. This contradicts the fact that $C_1$ must be acquainted with 4 people, so the case $k = 4$ is impossible.

Therefore, we have proven that the smallest number of people must be at least $n = 16$, which arises in the case $k = 5$. To see that this is indeed possible, consider the following diagram of acquaintances, with extra acquaintances between the people with the following labels.

\[
1 \leftrightarrow 8, 1 \leftrightarrow 9, 1 \leftrightarrow 10, 2 \leftrightarrow 6, 2 \leftrightarrow 7, 2 \leftrightarrow 10, 3 \leftrightarrow 5, 3 \leftrightarrow 7, 3 \leftrightarrow 9, 4 \leftrightarrow 5, \\
4 \leftrightarrow 6, 4 \leftrightarrow 8, 5 \leftrightarrow 10, 6 \leftrightarrow 9, 7 \leftrightarrow 8
\]