
(1) A specialist–mathematics class took a test worth 100 marks. Five of the students scored 100 and all of the students scored at least 60. If the average mark was 75, what is the least possible number of students in the class?

**Solution.** Let the number of students be $s$. If the lower scores are reduced to 60 then the average is no greater than 75. Hence

$$500 + 60(s - 5) \leq 75 \quad \Rightarrow \quad 200 \leq 15s \quad \Rightarrow \quad s > 13.$$ 

Try $s = 14$ with total score $14 \times 75 = 1050$. This can be achieved by

$$1050 = 5 \times 100 + 8 \times 60 + 1 \times 70.$$ 

(2) Joseph buys three shirts from a shop. At full price, each shirt costs a whole number of dollars and the total comes to $100. But the shirts are on sale. The cheapest shirt is $\frac{1}{5}$ off, the middle-priced shirt is $\frac{1}{6}$ off, the most expensive shirt is $\frac{1}{7}$ off, and the new discounted total is $84$. What are the possible prices of the three shirts?

**Solution.** Let the prices of the shirts from cheapest to most expensive be the integers $a, b$ and $c$. Then $a + b + c = 100$ and $\frac{1}{5}a + \frac{1}{6}b + \frac{1}{7}c = 16$. Furthermore, since the latter sum is an integer, $a = 5A$, $b = 6B$, $c = 7C$ where $A, B$ and $C$ are integers satisfying:

(i) $5A + 6B + 7C = 100$

(ii) $A + B + C = 16$

(iii) $5A \leq 6B \leq 7C$

(i) $- 16 \times$ (ii) $\Rightarrow C = A + 4$. Thus $2A + B = 12$ and

$$A = 1, 2, 3, 4, \quad B = 10, 8, 6, 4, \quad C = 5, 6, 7, 8$$

satisfy $5A \leq 6B$. To satisfy $6B \leq 7C$ we can only have

$$A = 3, 4, \quad B = 6, 4, \quad C = 7, 8.$$ 

Thus $(a, b, c) = (15, 36, 49)$ or $(20, 24, 56)$. 


(3) On each side of the shaded square of side length 2cm is a semi-circle. A rubber band is stretched around the four semi-circles as in the diagram. What is the length of the rubber band in this position?

Solution. Draw the broken lines as in the diagram. The rubber band consists of four quarter circle arcs and 4 line arcs. The circle arcs make up a full circle of radius 1cm and circumference $2\pi$cm. The line arcs each have the same length as the parallel broken line which is $\sqrt{2}$cm by Pythagoras theorem. Hence the length of the rubber band is $(2\pi + 4\sqrt{2})$cm.

(4) Let $S$ be the sum of the digits of all of the numbers from 1 to 2008. What is the remainder when $S$ is divided by 9?

Note that the sum of the digits of the first 20 numbers is

$$1 + 2 + ... + 9 + 1 + 0 + 1 + 1 + 1 + 2 + ... + 1 + 9 + 2 + 0 = 102.$$ 

Solution. Let $S$ be as above and $\hat{S}$ equal the sum of all of the numbers from 1 to 2008. The remainder when any number $N$ is divided by 9 is equal to the remainder when the sum of the digits of $N$ is divided by 9. Hence the remainder when $S$ is divided by 9 is equal to the remainder when $\hat{S}$ is divided by 9. Now $\hat{S} = 1004 \times 2009$. Summing digit shows that 1004 has 5 remainder and 2009 has 2 remainder, when divided by 9. Hence their product has $5 \times 2 = 10$ remainder which is the same as a remainder of 1. Hence when $S$ is divided by 9 the remainder is 1.
(5) Given any four-sided polygon, draw lines between the midpoints of it neighbouring sides to get a new four-sided polygon. Show that the new four-sided polygon has parallel sides.

Solution. In the triangle $ABC$ the line $PQ$ joins midpoints of two sides and hence it is parallel to the third side $AC$. Similarly the line $SR$ is parallel to $AC$, thus $PQ$ is parallel to $SR$ as required. A similar argument shows $PS$ is parallel to $QR$ proving that the new four-sided polygon has parallel sides.

(6) Two players play the following game. They take in turns rolling a standard dice (containing numbers 1 to 6.) They add the numbers from all previous rolls of both players. The first player to reach a total score of more than 5 loses the game. Here are a few games:

<table>
<thead>
<tr>
<th>Rolls</th>
<th>1, 2, 1, 3</th>
<th>6</th>
<th>5, 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>2nd player loses</td>
<td>1st player loses</td>
<td>2nd player loses</td>
<td></td>
</tr>
<tr>
<td>rolls</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1, 2, 1, 3</td>
<td>6</td>
<td>5, 1</td>
<td></td>
</tr>
<tr>
<td>1st player loses</td>
<td>2nd player loses</td>
<td></td>
<td></td>
</tr>
<tr>
<td>rolls</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1, 2, 1, 3</td>
<td>6</td>
<td>5, 1</td>
<td></td>
</tr>
<tr>
<td>1st player loses</td>
<td>2nd player loses</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Prove that the first player has a greater chance of winning the game.

Solution. The following table shows the probability that the 2nd player will lose after the second roll given each possible first roll by the 1st player.

<table>
<thead>
<tr>
<th>1st player roll</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>2nd player losing rolls</td>
<td>5, 6</td>
<td>4, 5, 6</td>
<td>3, 4, 5, 6</td>
<td>2, 3, 4, 5, 6</td>
<td>all</td>
<td></td>
</tr>
<tr>
<td>2nd player probability of losing</td>
<td>2/6</td>
<td>3/6</td>
<td>4/6</td>
<td>5/6</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Thus the probability that the 2nd player will lose is at least

$$\frac{1}{6} \times \frac{2}{6} + \frac{1}{6} \times \frac{3}{6} + \frac{1}{6} \times \frac{4}{6} + \frac{1}{6} \times \frac{5}{6} + \frac{1}{6} \times 1 + \frac{1}{6} \times 0 = \frac{20}{36} > \frac{1}{2}$$

so the 2nd player will lose with probability greater than 1/2.

(7) Find 20 consecutive integers $n+1, n+2, \ldots, n+20$ less than 10000 each with no prime factor greater than 18.

Solution. There are no such integers since any 20 consecutive integers must contain a number with a factor of 19.