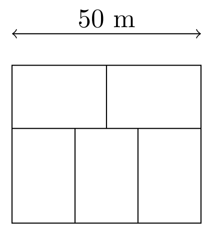
1. Olympic glory. Grant Hackett recently visited Beijing to check out the "Water Cube". This is the futuristic looking aquatic centre where Grant hopes to win his third Olympic gold medal in the 1500 metres freestyle.
Grant, who is a bit of a maths whiz, observed the curious fact that the pool can be partitioned into 5 congruent (equal-sized) smaller pools as shown in the figure below.



The length of the pool is 50 metres, and at the Olympics there will be 10 lanes (only 8 in the centre will be used). Using this information Grant was able to work out the width of each lane. Can you too?

Solution
Let each of the small rectangles have length x and width y (measured in metres). From the picture it follows that $2x = 3y = 50$. Hence $x = 25$ and $y = 50/3 = 16\frac{2}{3}$. From the picture it also follows that the width of the pool is $x+y = 25+16\frac{2}{3} = 41\frac{2}{3}$. Dividing this by 10 to find the width of each lane gives

$$\text{width of each lane is } \frac{25}{6} = 4\frac{1}{6} \text{ metres}$$

which is more than enough to hold Grant's huge shoulders.

2. BHP. For many years now BHP Billiton has supported the maths competition. Interestingly, the total prize money (in dollars) for this year's Junior competition is exactly the same as the largest possible outcome of

$$\mathrm{B} + \mathrm{H} + \mathrm{P}$$

where B, H and P are positive whole numbers, with *no two of them equal to each other*, such that

$$\mathrm{B} \times \mathrm{H} \times \mathrm{P} = 2008.$$

Find the total prize money.

Solution
The number $2008 = 2 \times 2 \times 2 \times 251$. (This is known as the prime factorisation.) To make the sum of B, H and P as large as possible we should thus take $\mathrm{B} = 1$, $\mathrm{H} = 2$ and $\mathrm{P} = 2 \times 2 \times 251 = 1004$ (or any reordering of B, H and P). The total prize money is therefore

$$1004 + 2 + 1 = 1007.$$

3. Fractured fractions. The ancient Egyptians were pretty advanced in their mathematics. Dealing with fractions was not a problem for them despite the curious custom of writing fractions in the form

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \cdots,$$

where $a, b, c, \ldots$ are whole numbers, *with no two of them equal to each other.*

For example, instead of writing

$$\frac{3}{8}$$

they would write

$$\frac{1}{4}+\frac{1}{8} \qquad \text{or} \qquad \frac{1}{3}+\frac{1}{24}$$

or, if they wanted to impress the ladies (mathematics was strictly men's business in ancient times),

$$\frac{1}{3}+\frac{1}{33}+\frac{1}{88}.$$

Write

$$\frac{9}{13}$$

in ancient Egyptian form.

Solution

There are many strategies to achieve this (and also many different answers). Whatever the method, there will always be one number that is going to be the largest denominator. This largest denominator is always a multiple of the denominator of the fraction of interest. For example, in the three different ways we have written $3/8$ we have 8, $24 = 3 \times 8$ and $88 = 11 \times 8$ as largest denominators. One method is to guess (there are methods to help with this) a largest denominator and to see if this simplifies the fraction. For example, trying 88 in the case of $3/8$ leaves a remainder of $3/8 - 1/88 = 32/88 = 4/11$ which is actually more complicated than $3/8$ and so does not lead to the easiest solution (hence guessing 8 or 24 works better). In the current problem we might first guess 13 but this leads to $9/13 - 1/13 = 8/13$ which does not appear that much simpler. Trying 26 gives $9/13 - 1/26 = 17/26$ which again is not particularly nice. Our third try of 39 works well however: $9/13 - 1/39 = 26/39 = 2/3$. Now we still need to write $2/3$ in Egyptian form but the guess of 6 now solves the problem $2/3 - 1/6 = 3/6 = 1/2$ so that

$$\frac{9}{13}=\frac{1}{2}+\frac{1}{6}+\frac{1}{39}.$$

4. Money matters. At an initial cost of only \$100, Ms Moneypenny is offered to join the following get-rich-quick scheme. Given a 50 cent coin she can trade it (at no cost) for five 10 cent coins, and given a 10 cent coin she can trade it (again at no cost!) for five 50 cent coins. She is given one 10 cent coin to start out her money-making, but the rule is that she loses all if at the end of the day she does not have an equal amount of money in 10 cent coins and in 50 cent coins.

For example, if Ms Moneypenny ends up with 5000 coins of 10 and 1000 coins of 50 she can keep it all, but if she ends up with 5000 coins of 10 and 999 coins of 50 she loses all.

Can Ms Moneypenny make her fortune by participating in the scheme? (Fully explain your answer.)

Solution

The answer is no. Get-quick-rich schemes never work.

To have an equal amount of money in 50s and 10s, Ms Moneypenny needs five 10 cent coins for each 50 cent coin. Since $5 + 1 = 6$ this means she needs an even number of coins. But at each trading step she hands in one coin and gains 5 coins

so that the number of coins goes up by 4 at each trade. Starting with one coin this means that the total number of coins she will have is always an odd number.

5. Chess competition. In an inter-school chess competition each participating school enters a team of two players. Each participant plays all other players except their team-mate. The winner of each game receives one point and the loser gets nothing. If a game is drawn both players receive half a point.
At the end of the competition your school has a total winning score of 39 points, and all other schools have an equal number of points.
How many schools participated in the competition?

Solution
Let n denote the number of participating schools. Hence there are $2n$ players. Since each player plays $2(n-1)$ games the total number of games played is $2n \times 2(n-1) : 2 = 2n(n-1)$. In each game one point is awarded so that the total number of points won by all schools combined is exactly $2n(n-1)$. Of course, your school has won 39 out of this total.

Let C be the common score of all other schools. Since there are $n-1$ such schools the total number of point won by all other schools is $C(n-1)$. Therefore

$$2n(n-1) = 39 + C(n-1).$$

This can also be written as

$$39 = 2n(n-1) - C(n-1) = (2n-C)(n-1)$$

or as

$$3 \times 13 = (2n-C)(n-1).$$

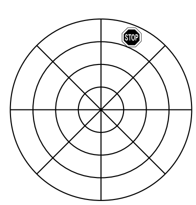
Since 3 and 13 are primes the only four possibilities are

(1) $2n - C = 1$ and $n - 1 = 39$, which implies $n = 40$ and $C = 79$,
(2) $2n - C = 39$ and $n - 1 = 1$, which implies $n = 2$ and $C = -35$,
(3) $2n - C = 3$ and $n - 1 = 13$, which implies $n = 14$ and $C = 25$,
(4) $2n - C = 13$ and $n - 1 = 3$, which implies $n = 4$ and $C = -5$.

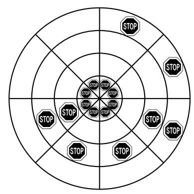
The solutions (2) and (4) do not make sense because C is negative. The solution (1) contradicts the fact that your school had a winning score of 39 points (since it claims the other schools all have a score of 79). Therefore (3) must be correct and $n = 14$.

Note that this is certainly possible; if you win all your games and your team-mate draws all his/her games, and all games between the other schools are drawn, then your school gets $26 + 26 \times 1/2 = 26 + 13 = 39$ and all other schools get $2 \times 25 \times 1/2 = 25$.

6. Stop it. The game of *Stop it* is played by two players on a board containing 32 "tiles". At the start of the game there is one stop sign on the board in the position shown below:

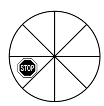


When it is a player's turn, they can make three moves: (1) put one stop-sign on the tile next to the last-placed sign, moving in clockwise direction; (2) put two stop-signs on the two tiles next to the last-placed sign, moving in clockwise direction; (3) put one stop-sign on the tile next to the last placed sign, moving one step closer to the centre of the board.
Each tile can contain only one stop-sign, and the first player who cannot put another stop-sign loses.
A typical example of the board at the end of the game looks like

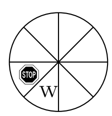


Show that if someone as smart as you starts the game, they will always win.

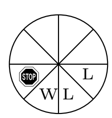
Solution
To find a winning strategy let us first show that the player who enters the inner-most circle first will lose. The first time the inner-most circle is entered, it looks something like



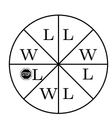
Of course we do not exactly know which of the 8 fields will have the stop sign but this does not matter for the argument. Given the above picture and the fact that we move clockwise it is clear that being able to place the final stop sign on the field labelled W means you have won; your opponent is the first player who cannot go any further.



This in turn means that the player who occupies either of the two fields prior to W loses, because the next player can then jump one or two fields ahead to occupy W. Hence we get

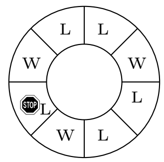


We can continue this reasoning backwards and determine winning and losing fields:



It is important to note that the field that was first entered is a losing field. Hence you will win if you make your opponent enter the inner circle first: from then on you can make sure to only cover W(inning) fields.

But it is now clear how to make your opponent enter the inner-most circle first; make them enter the ring (mathematician say annulus) surrounding the inner-most circle first: this again is an L field (marked with a) and you can make sure only to cover W fields. Of course your opponent may decide to not complete all of the annulus surrounding the inner-most circle and jump to the inner-most circle ahead of time. Bad luck, we have already seen this will not save them.



By a repeat of the above, it follows that at each stage you must let your opponent be the one to make a move closer to the center. This can be done because the field immediately following the initial stop sign is a W field: