Q1. A 100m long walkway moves at a constant speed of 6 metres per second. Amelia steps on to the start of the walkway at time $t = 0$ and stands there, letting it move her at 6 m/s. Bob steps onto the walkway 2 seconds later and strolls forward at a constant speed of 4m/s. Two seconds later Catherine reaches the start of the walkway but decides not to use it, and instead strolls briskly alongside the walkway (i.e. not on it) at a constant speed of 8 m/s. At time $t = T$, one of the three is exactly halfway between the other two. Who is this person? Find T, and the distance L between the start of the walkway and the middle person.

Solution. Implicitly $T > 4$, otherwise we don't know where C is. Then at time T, their respective positions are $6T, 10(T-2)$ and $8(T-4)$.

- Case: A in middle. Then
$$2 \times 6T = 10(T-2) + 8(T-4) \Leftrightarrow T = \frac{26}{3},$$
and $L = 6T = 52$ (metres).
- Case: B in middle. Then
$$2 \times 10(T-2) = 6T + 8(T-4) \Leftrightarrow T = \frac{4}{3},$$
which contradicts the condition $T > 4$.
- Case: C in middle. Then
$$16t - 64 = 16t - 20,$$
contradiction.

Hence it is Amelia who is in them middle, at time $T = \frac{26}{3}$, and she is 52 metres along the runway at this time.

Q2. In an equilateral triangle ABC, the point P is on AB so that $AP = AB/3$, the point Q is on BC so that $BQ = BC/3$, and the point R is on CA so that $CR = CA/3$. The lines CP, AQ and BR enclose a triangle. Find the ratio of the area of this triangle to that of ABC.

Solution. Let CP and BR meet at X, CP and AQ at Y, BR and AQ at Z. Let $|\cdot|$ denote area. Moreover, let $|APY| = |BQZ| = |CRX| = m$, $|BYZ| = n$.

The key idea is that triangles with the same height have areas in the ratio of their bases. As $BP = 2AP$, we have
$$|BPY| = 2|APY| = 2m.$$

Now
$$|AYXR| = |CXZQ| = |BZYP| = 2m + n.$$

By the same argument,
$$|CQY| = 2|BQY| = 2m + 2n \therefore |XYZ| = n.$$

Similarly,

$$\begin{aligned} |CQA| &= |BQA| \\ 2(4m+n) &= 5m+3n \\ n &= 3m. \end{aligned}$$

Replacing n with $3m$ gives $|XYZ| = 3m$ and $|ABC| = 21m$, so the desired ratio is $\frac{1}{7}$.

Q3. Consider a $2n \times 2n$ chessboard, as usual constructed with alternating black and white squares. Now remove any two squares, one black and one white. After the removal of the two squares, is it possible to cover the board with 2×1 dominoes? (The answer is required for all positive integer values of n.)

Solution. Consider the smallest rectangle containing the two removed squares. Since the squares are of different colour, the dimensions of this rectangle have different parity. Without loss of generality, say it has an even number of columns (and an odd number of rows). The columns to the right of this rectangle can be tiled with vertical dominoes (since the square has an even number of rows), whilst the remaining 'outer' squares can be tiled with horizontal dominoes (since the rectangle has an even number of columns). Now it remains to tile the rectangle, apart from the removed squares.

If the removed squares lie in the same row or column, it's easy to check that this rectangle can be tiled, so suppose that they don't. Since the squares directly above the lower removed square with vertical dominoes (possible since the rectangle has an odd number of rows, and the bottom square in this column has been removed). Similarly, tile the squares directly below the upper removed square with vertical dominoes. This leaves a complete rectangle, with an even number of columns, to be tiled, which can be achieved with horizontal dominoes.

Q4. If we move the least significant digit of an integer to the most significant position, we call this a 1-permutation. For example, 3124 would become 4312 under a 1-permutation. Find an integer n which, when multiplied by 4 becomes equal to itself after a 1-permutation. That is to say, $4n$ is equal to the 1-permutation of n.

Solution. $128205 \times 4 = 512820$, so $n = 128205$ is one possible solution (there are others).

Q5. Three ordered numbers $a \leq b \leq c$ can form the sides of a triangle if and only if $a + b > c$. A set of positive numbers is said to possess the *triangle property* if there are three distinct elements that can form the sides of a triangle. Consider sets $\{4, 5, 6, \ldots, n\}$ of consecutive positive integers, all of whose ten-element subsets have the triangle property. (This means that one can choose any 10 elements—they do not need to be consecutive). What is the largest possible value of n?

Solution. Suppose one of these subsets does not possess the triangle property. Denote its elements by $a_1, a_2, \ldots, a_n$, in increasing order. $a_1 \geq 4, a_2 \geq a_1 + 1 \geq 5$. Now

$$\begin{aligned} a_3 &\geq a_1 + a_2 \geq 9, \\ a_4 &\geq a_2 + a_3 \geq 14. \end{aligned}$$

Continuing, we see that $a_5 \geq 23, a_6 \geq 37, a_7 \geq 60, a_8 \geq 97, a_9 \geq 157, n \geq a_{10} \geq 254$. Therefore all 10-element subsets possess the triangle property whenever $n \leq 253$ (the *contrapositive* statement).

If $n \geq 253$, then $\{4, 5, 9, 14, 23, 37, 60, 97, 157, 254\}$ is a 10-element subset not possessing the triangle property. Hence 253 is the largest satisfactory value of n.

Q6. Let f be a function on the set of all natural numbers $\{1, 2, 3, \ldots\}$. If $f(1) = 1$, $f(2n) = f(n)$ and $f(2n + 1) = f(2n) + 1$, (a) calculate the maximum value N of $f(n)$ if $1 \leq n \leq 2009$, and (b) find all values of n with $1 \leq n \leq 2009$, such that $f(n) = N$.

Solution.
If $f(n)$ is the sum of digits of n in binary, we see that this satisfies the conditions. To see that this is the only function satisfying the conditions, observe that any function value is determined by the function's value on lower inputs (more formally, we would do a proof by *strong induction*), so there's a unique such function, which we've chanced upon finding.

If $f(n) \geq 11$, then $n \geq 1 + 2 + \ldots 2^{10} = 2^{11} - 1 = 2047 > 2009$. Hence $N \leq 10$. If $f(n) = 10, n \leq 2009$, then n is an 11-digit binary number with exactly one zero (maybe at the start). In other words,

$$n = 2047 - 2^k,$$

for some integer k.

The condition $1 \leq n \leq 2009$ now gives $6 \leq k \leq 10$, so we get 5 solutions: 1023, 1535, 1791, 1919, 1983, and it's easy to see that $f(n) = 10$ for all of these, so $N = 10$.

Q7. A group of 37 friends decide to play a game of Australian Rules football. There will be 18 players in each team, and one referee. Assume for simplicity that each person is of positive integer weight (a whole number of kilograms). To ensure physical equality, they split into two teams of equal total weight. To their surprise, they find that this can be done no matter who they choose as referee. Prove that all 37 friends must have the same weight.

Solution. Call the condition the *referee property.*

Subtract the lightest player's weight from every player's weight, we have 37 non-negative integers with the referee property.

Let a, b be two of these numbers, and let T be the sum of all 37 of them. $T - a$ and $T - b$ are even, so a and b have the same parity. Since a and b were chosen arbitrarily, it follows that all 37 numbers have the same parity. As (at least) one of them is 0, it follows that they're all even.

Suppose one of the numbers is positive, call it n. Dividing all of the numbers by 2 preserves the referee property, and at least one of them will still be zero. Do this k times, where $2^k > n$, and n becomes $\frac{n}{2^k}$, which lies strictly between 0 and 1, contradiction.

Thus none of the numbers were originally positive, i.e. they were all 0, i.e. everyone had the same weight at the start.

Q8. Determine whether there exists a positive integer n such that the sum of the digits of n^2 is 2008. If so, find n, and comment on whether n is unique.

Solution. Let $k \geq 1$. Now

$$(10^k - 2)^2 = 100^k - 4 \times 10^k + 4 = 4 + 10^k(10^k - 4),$$

which has sum of digits equal to $4 + 9(k - 1) + 6 = 9k + 1$.

Observe that $k = 223$ gives $9k + 1 = 2008$. Hence $n = 10^{223} - 2$ is such that n^2 has 2008 as its sum of digits. It's not unique, since adding 0s to the end doesn't affect the outcome.

There are other solutions; for instance try $39 \dots 95$.

Comments

- We expected students to prove that there wasn't more than one solution in Q1.
- Many students misinterpreted Q2. Read the question carefully.
- Many of this year's questions involved trial and error. If the question asks to 'find an n such that...', it is necessary to prove that your n satisfies the conditions. It isn't necessary to explain how you got it, so long as you've properly explained why it works.
- The solutions presented were deemed to be simplest. Sometimes there were other methods: Q2 was solved differently using similar triangles and/or trigonometry and/or Menalaus' theorem! Meanwhile, Q3 was nicely proven by induction.