These questions are designed to test your ability to analyse a problem and to express yourself clearly and accurately. The following suggestions are made for your guidance:

(1) Great weight will be attached by the examiners to the method of presentation of a solution. Candidates should state as clearly as they can the reasoning by which they arrived at their results. In addition, more credit will be given for an elegant than for a clumsy solution.

(2) The six questions are not of equal length or difficulty. Generally, the later questions are more difficult than the earlier questions.

(3) It may be necessary to spend considerable time on a problem before any real progress is made.

(4) You may need to do considerable rough work but you should then write out your final solution neatly, stating your arguments carefully.

(5) Credit will be given for partial solutions; however a good answer to one question will normally gain you more credit than sketchy attempts at several questions.

Textbooks, electronic calculators and computers are NOT allowed. Otherwise normal examination conditions apply.
1. **Awesome foursome.** Fill out the following $4 \times 4$ square such that each row and column contains the numbers 1, 2, 3 and 4 exactly once, and such that the sum of the numbers in each region bounded by solid lines is as indicated by the number shown in the top-right corner.

   ![4x4 Grid](image)

2. **Masterpieces from Paris.** The recently closed exhibition “Masterpieces in Paris”, held at the National Gallery of Australia in Canberra and showing paintings by masters such as van Gogh, Gauguin and Cézanne, charged a $30 admission fee. Being less cultured and more stingy than Victorians, the people of Canberra thought this too expensive and visitor numbers were disappointingly low. After reducing the admission fee, the number of daily visitors went up by 50% while the museum’s earnings went up by 25%.

   What was the new admission fee for the exhibition?

3. **Grumpy.** Snow White is having peas for dinner. Being a princess, she does not like peas very much, so instead of finishing her meal, she starts counting the number of peas left on her plate. She asks the dwarfs to make guesses about this number. Sleepy guesses that the number of peas is a multiple of 10 (i.e., the number of peas is one of 10, 20, 30,...). Dopey guesses that the number of peas is a multiple of 12, Happy guesses that the number of peas is a multiple of 15, Sneezy guesses that the number of peas is a multiple of 18, and, finally, Doc guesses that the number of peas is a multiple of 30. Grumpy, who does not like games, spoils it for everyone. He has quickly counted the number of peas on Snow White’s plate and says “only two of you guessed correctly”.

   Which of the two dwarfs made a correct guess?

4. **Doing a Barnaby.** During his short stint as Shadow Finance Minister, Barnaby Joyce displayed a curious grasp of numbers. One of his most infamous exclamations was “All this billions, quillions, Brazilians.”

   Less well known was his tax-reform proposal, recommending that if you earn $x$ per day, you are taxed $x\%$ of that amount (with the understanding that you pay 100% tax if you earn $100 or more per day.

   Under Senator Barnaby’s plan, which amount $x$ gives you the highest earnings after tax has been taken out?
5. **Year of the Tiger.** Exactly 100 people sit around a very large dinner table at the world’s most expensive restaurant, El Bulli in Spain. Every diner may order any number of dishes with the one condition: the combined bill of a diner and his/her two neighbours must exactly be 2010 Euros.
Show that each diner must spend exactly 670 Euros.

6. **Hospital beds.** When Kevin Rudd recently visited the Royal Melbourne Children’s hospital to promote his health-care reform plans, one of the nurses tried to test Kevin’s knowledge of hospital bed numbers. She told him there are two types of hospital beds, the cheap ones, which desperately need replacement, with only 4 legs, and the expensive, much more comfortable ones, with a 5th leg in the middle for extra support. She also told Kevin that in ward X there are **beds** of **both types** with a total of $n$ legs and asks him to figure out how many beds there are. After a quick phone call to Minister for Education Julia Gillard, Kevin triumphantly declared

“\[correct number deleted\] beds in the ward. In fact $n$ is the smallest number such that, had there been $n+1$ legs in the ward, Julia, errh, I mean I, would not have been able to tell you the number of beds”.

Find the number of beds in ward X.
Awesome foursome. As a first step, note that the solitary square in the top row must contain the number 1. Note that the three-square region in the bottom row must sum to 9. Since $9 = 2 + 3 + 4$, the square in the bottom right-hand corner must contain the number 1:

```
  1
 / \
1 3 5
 /   \
4 9 1
```

We can now put a 4 above the bottom 1 to form $1 + 4 = 5$:

```
  1
 / \
1 3 5
 /   \
4 9 1
```

Since $3 = 1 + 2$, and the second column already contains a 1, we can now complete the third row as follows:

```
  1
 / \
1 2 3 4
 /   \
4 9 1
```

We can now put the remaining 1 in row 2:

```
  1
 / \
1 2 3 4
 /   \
1 9 1
```

Finally, we need to add 2, 3, 4 in each of the three remaining regions to sum to 9. Firstly, since the fourth column already has a 4, we may put a 4 immediately to the right of the 1 in the first row, which only leaves one possibility for a 2 in the third column. Hence
The rest merely comes down to completing all the squares, which can be done in more ways than one:

\[
\begin{array}{cccc}
2 & 1 & 4 & 3 \\
3 & 4 & 1 & 2 \\
1 & 2 & 3 & 4 \\
4 & 3 & 2 & 1 \\
\end{array}
\]

\[
\begin{array}{cccc}
2 & 1 & 4 & 3 \\
3 & 4 & 1 & 2 \\
1 & 2 & 3 & 4 \\
4 & 3 & 2 & 1 \\
\end{array}
\]

\[
\begin{array}{cccc}
3 & 1 & 4 & 2 \\
2 & 4 & 1 & 3 \\
1 & 2 & 3 & 4 \\
4 & 3 & 2 & 1 \\
\end{array}
\]

Masterpieces from Paris. Let \( A_0, N_0 \) and \( E_0 \) be the admission fee (in dollars), number of daily visitors, and daily earnings (in dollars) respectively, before the prices were lowered. Then \( A_0 = 30 \) and \( E_0 = A_0N_0 \) (earnings=admission fee times number of visitors).

Let \( A_1, N_1 \) and \( E_1 \) be the admission fee, number of daily visitors, and daily earnings respectively, after the prices were lowered. Then \( E_1 = A_1N_1, \quad N_1 = \frac{3}{2}N_0 \) and \( E_1 = \frac{5}{4}E_0 \).

Hence

\[
A_1 = \frac{E_1}{N_1} = \frac{\frac{5}{4}E_0}{\frac{3}{2}N_0} = \frac{5}{6} \frac{E_0}{N_0}.
\]

But \( E_0/N_0 = A_0 = 30 \) so that

\[
A_1 = \frac{5}{6} \times 30 = 25.
\]

The new admission price was therefore $25.

Grumpy, long solution. Doc cannot be right because then Sleepy and Happy would also have to be right, making for three right guesses; the number of peas is not a multiple of 30.

At least one of Dopey, Happy and Sneezy must have guessed correctly. But their guesses (a multiple of 12, 15 and 18, respectively) are all multiples of 3. Hence the number of peas must be a multiple of 3 as well. But this means that Sleepy cannot be right, because a multiple of 10 that is also a multiple of 3 is a multiple of 30, which was ruled out earlier.

So we are down to Dopey, Happy and Sneezy. But 15 = 3 x 5 and 18 = 3 x 3 x 2 so that any multiple of 15 and 18 must be a multiple of 3 x 3 x 2 x 5 = 90, which is a multiple of 10 (and 30). This we already know to be impossible, so only one of Happy and Sneezy is right. By the same reasoning, only one of Happy and Dopey is right: 12 = 2 x 2 x 3 and 15 = 3 x 5 so that any multiple of 10 and 15 must be a multiple of 2 x 2 x 3 x 5 = 60, which is again a multiple of 10 (and 30).

In conclusion, both Dopey and Sneezy were right, and Snow White had a multiple of 36 peas on her plate (but of course not any multiple of 36, because 5 times 36, for example, is also a multiple of 10, 15 and 30).
Grumpy, short solution (by “almost” cheating). We know that exactly two dwarfs guessed correctly. Now make the inspired guess that these two were Dopey and Sneezy. Then the number of peas must have been a multiple of 12 as well as a multiple of 18, i.e., the number of peas must have been a multiple of 36. Since it is possible to find multiples of 36 (for example 36, 72, 108) that are not multiples of 10 and 15, we can never rule out that Dopey and Sneezy had been right. So the answer must have been Dopey and Sneezy . . .

Doing a Barnaby. If you earn $x$ (in dollars) and pay $y\%$ tax, this means you pay the tax office

$$
\frac{y}{100} \times x = \frac{xy}{100}
$$

in tax. In this particular question $y$ is actually equal to $x$, so that you pay the tax office $x^2/100$. If you earn nothing at all, then you pay no tax at all and your after-tax earnings come down to the meagre amount of 0. If you earn 100 you pay $100^2/100 = 100$ in tax and again you are left empty handed. If you earn $0 < x < 100$ then your earnings minus what you pay the tax office is given by

$$
\text{after-tax earnings (in dollars)} = x - \frac{x^2}{100} = \frac{x(100-x)}{100}.
$$

We have already seen that this is 0 for $x = 0$ and for $x = 100$. If you plot the above you get the graph

![Graph showing the after-tax earnings for different values of x, with the maximum at x = 50.](image)

where the horizontal axis gives your before-tax earnings (i.e., $x$) and the vertical axis your after-tax earnings. The graph has its maximum at $x = 50$, in which case your after-tax earnings are

$$
\frac{50(100 - 50)}{100} = \frac{50 \times 50}{100} = 5 \times 5 = 25.
$$

Remark: For those wishing a more rigorous solution (i.e., not depending on the sketching of a graph), there are (at least) two ways to go about this, neither of which are expected at junior secondary school level.

1. The function $f(x) = x(100-x)/100$ describes a downward parabola. Every such parabola has exactly one maximum, which can be found by solving the equation $f'(x) = 1 - x/50 = 0$. This gives $x = 50$.

2. The function $f(x) = x(100-x)/100$ describes a downward parabola. Since $f(x) = f(100-x)$ its maximum must be at $x = 50$. 
**Year of the Tiger.** Label the diners from 1 to 100. Hence diner \( n \) sits next to diners \( n - 1 \) and \( n + 1 \) with the exception of 1 (who sits next to 2 and 100) and 100 (who sits next to 99 and 1).

Let diners 1, 2 and 3 spend \( a \) Euros, \( b \) Euros and \( c \) Euros respectively. Since the total amount spent by the first three diners is \( a + b + c \) the amount spent by any triple of consecutive (in number) diners must also be \( a + b + c \). For example diners 33, 34 and 35 must also spend \( a + b + c \) Euros in total, and so must 99, 100 and 1. (We in fact know that \( a + b + c = 2010 \) but this will only become important later).

Since diners 2, 3 and 4 must spend a total of \( a + b + c \) Euros and 2 is spending \( b \) and 3 is spending \( c \) Euros, diner 4 must be spending \( a \) Euros.

Since diners 3, 4 and 5 must spend a total of \( a + b + c \) Euros and 3 is spending \( c \) and 4 is spending \( a \) Euros, diner 5 must be spending \( b \) Euros.

Continuing this reasoning leads to the conclusion that diners with numbers given by

\[
1, 4, 7, 10, 13, 16, \ldots, 100
\]

must all spend \( a \) Euros. (Note that 100 indeed belongs to the above list of numbers because the list is given by all numbers of the form \( 1 + 3n \) and \( 100 = 1 + 3 \times 33 \).)

But this means that diner 1 who comes after diner 100, and who we said was spending \( a \) Euros, must in fact be spending \( b \) Euros. The inevitable conclusion is that \( a = b \). So now we know that diners 100, 1 and 2 are all spending \( a \) Euros, i.e., they all spend the same amount, namely \( 2010/3 = 670 \). Obviously all other diners are now forced to also spend exactly this amount.

**Hospital beds.** We will first try to figure out the smallest number of legs for which there is more than one possible number of beds. For example if there would have been 34 legs then we could have had 7 beds (1 bed of 4 legs and 6 beds of five legs) or 8 beds (6 beds of 4 legs and 2 beds of 5 legs). As we shall see 34 is not the smallest number that allows for more than one “solution”.

Let \( A \) be the number of beds with 4 legs and \( B \) be the number of beds with 5 legs. Then the total number of legs is \( 4A + 5B \). If this number of legs allows for a second “solution” we must have numbers \( A' \) and \( B' \) such that

\[
4A + 5B = 4A' + 5B'
\]

and such that \( A' \neq A \). Without loss of generality we may assume that \( A' > A \) (so that \( B' < B \)). We write the above equation as

\[
4(A' - A) = 5(B - B').
\]

Since the left is a multiple of 4, the right must also be a multiple of 4. Similarly, since the right is a multiple of 5, the left must also be a multiple of 5. Hence \( A' - A = 5k \) and \( B - B' = 4k \) where \( k \) is a positive integer. Because \( A, A', B, B' \) are all positive the “smallest solution” to the above is \( k = 1 \) and \( A = 1, B' = 1 \) so that \( A' = 6 \) and \( B = 5 \). We can check that \( 4A + 5B = 4 + 25 = 29 \) and \( 4A' + 5B' = 24 + 5 = 29 \). Hence the smallest number of legs for which we do not get more than one solution is 28, which can only come from 4 beds with 5
legs and 2 beds with five legs. This is the number the nurse gave Kevin Rudd, and the total number of beds in the ward is therefore 6.