THE UNIVERSITY OF MELBOURNE/STATISTICAL CONSULTING CENTRE SCHOOL MATHEMATICS COMPETITION, 2011

PRELIMINARY SOLUTIONS -JUNIOR DIVISION

(These solutions may be updated with improved versions, and/or comments added after marking.)

1. The Bev. Frendan Bevola, desperate for some extra cash, borrows money from his estranged wife Alex and decides to try his luck at Crown Casino. Playing roulette he wins his first bet, doubling the money he borrowed from Alex. He isn't so lucky in his next bet, losing $10,000. He then has another win, once again doubling his money, and another loss, costing him $10,000. After doubling his money for a third time Bev loses $10,000 for a final time, leaving him without a single cent.
How much money did The Bev borrow from Alex?

Solution. The Bev ends up losing all of his money. Hence, before his final loss, he had exactly $10,000. He got to this by doubling his money so he had $5,000 before this doubling took place. Continuing to work backwards, leads to

$$0 \xleftarrow{\text{third loss}} 10,000 \xleftarrow{\text{third win}} 5,000 \xleftarrow{\text{second loss}} 15,000 \xleftarrow{\text{second win}} 7,500 \xleftarrow{\text{first loss}} 17,500 \xleftarrow{\text{first win}} 8,750$$

Bevola thus borrowed $8,750 from Alex.

2. Year of the rabbit. It is a little-known fact that former Melbourne Lord Mayor John So started his professional career as a teacher at Fitzroy High School. Thanks to his legendary maths puzzles he was voted most popular teacher at the school—a feat he later reprised in his role as Lord Mayor, when he became the world's most popular Mayor. From the school archives the competition organisers have unearthed the following puzzle, which John So prepared to celebrate Chinese New Year.
Consider a triangle consisting of ten numbers, four in the top row, three in the second row, two in the third row and one in the bottom row, such that each number in the bottom three rows is formed by summing up the two numbers directly above it. An example of such a triangle is

$$\begin{array}{ccccccc} \boxed{10} & & \boxed{3} & & \boxed{1} & & \boxed{8} \\ & \boxed{13} & & \boxed{4} & & \boxed{9} & \\ & & \boxed{17} & & \boxed{13} & & \\ & & & \boxed{30} & & & \end{array}$$

Complete the following triangle according to the same pattern:

$$\begin{array}{ccccccc}
\boxed{20} & & \boxed{} & & \boxed{} & & \boxed{11} \\
& \boxed{} & & \boxed{} & & \boxed{} & \\
& & \boxed{} & & \boxed{} & & \\
& & & \boxed{2011} & & &
\end{array}$$

Hint: There is more than one possible solution.

Solution. The most general triangle is

$$\begin{array}{ccccccc}
\boxed{a} & & \boxed{b} & & \boxed{c} & & \boxed{d} \\
& \boxed{a+b} & & \boxed{b+c} & & \boxed{c+d} & \\
& & \boxed{a+2b+c} & & \boxed{b+2c+d} & & \\
& & & \boxed{a+3b+3c+d} & & &
\end{array}$$

If $a = 20$ and $d = 11$ this implies that $2011 = a + 3b + 3c + d = 3b + 3c + 31$. Hence $3b + 3c = 2011 - 31 = 1980$ so that $b + c = 660$. As long as you choose b and c such that they sum up to 660 you get a correct solution. For example, $b = 1$ and $c = 559$:

$$\begin{array}{ccccccc}
\boxed{20} & & \boxed{1} & & \boxed{559} & & \boxed{11} \\
& \boxed{21} & & \boxed{660} & & \boxed{570} & \\
& & \boxed{681} & & \boxed{1330} & & \\
& & & \boxed{2011} & & &
\end{array}$$

3. Lady Gaga. Lady Gaga's soon-to-be-released single "Mathematical Oddity" is a compilation of five of her best-known songs. In the table below you can find the duration (in seconds) of each of the five songs that make up "Mathematical Oddity".

	Just Dance	Poker Face	LoveGame	Paparazzi	Bad Romance
Duration of song	93	84	82	78	73

The mathematical oddity of Lady Gaga's new single is that if you play the first 2 songs only, then the average length of these two songs is a whole number of seconds. If that's not odd enough, if you play the first 3 songs only, then the average length of these three songs is also a whole number of seconds. Even more bizarrely, if you play the first 4 songs only, then the average length of these four songs is also a whole number of seconds. Of course, if you play all 5 songs, the average length is also a whole number, since $(93 + 84 + 82 + 78 + 73)/5 = 82$.

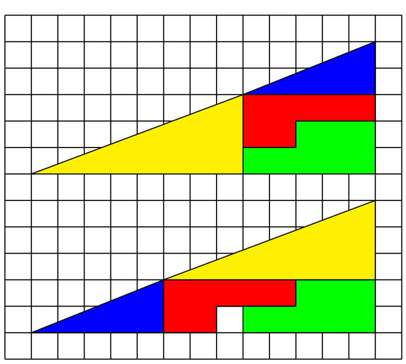
Which of the 5 songs is played last on “Mathematical Oddity”?

Solution. There are five songs, two of odd length and three of even length. After playing four of the five, the total length (in seconds) must be a multiple of four. Hence, among the first four songs there must be two odds and two evens. Because $73 + 93 = 166$ these two evens must add up to make a multiple of two, but not a multiple of four. Hence 84 is one of the evens and either 78 or 82 is the other even. In other words, there are two possibilities: The first four songs (in no particular order) have lengths $73, 78, 84, 93$ (so that the average length is 82) or the first four songs have lengths $73, 82, 84, 93$ (so that the average length is 83).
By a similar kind of reasoning, the first two songs must either both be odd or both be even.
Let’s assume we start with two odds, i.e., with 73 and 93. The order in which these are played is irrelevant; their average is 83 which is a whole number. We now need to pick a third songs so that the first three songs sum to a multiple of three. Since 93 itself is a multiple of three we must find an even songs such that its sum with 73 gives a multiple of three. But $73 + 78 = 151$, $73+82 = 155$, $73+84 = 157$, none of which is a multiple of three. (A quick way to check a number is a multiple of three is by checking that the sum of its digits is a multiple of three: $1 + 5 + 1 = 7$ is not a multiple of three, $1 + 5 + 5 = 11$ is not a multiple of three and $1 + 5 + 7 = 13$ is not a multiple of three.) Hence we must start with two evens.
We are now left with only a few cases to check. If (the lengths) of the first four songs are $73, 82, 84, 93$ then 82 and 84 are played first. But $82 + 84 + 73 = 239$ is not a multiple of three and $82 + 84 + 93 = 259$ is not a multiple of three. Therefore the first four songs are $73, 78, 84, 93$ with 78 and 84 played first. Since $73 + 78 + 84 = 235$ is not a multiple of three but $78 + 84 + 93 = 255$ is, we know that “Mathematical Oddity” is played in the following order:

$$78,\ 84,\ 93,\ 73,\ 82 \qquad \text{or} \qquad 84,\ 78,\ 93,\ 73,\ 82.$$

Hence LoveGame is played last.

4. Never trust your teacher. Your maths teacher is extremely excited. Using four different shapes, as shown in the two pictures below he believes to have irrefutably proven that $32.5 = 31.5$.



Area coloured by shapes $= \frac{13\times 5}{2} = 32.5$

Area coloured by shapes $= \frac{13\times 5}{2} - 1 = 31.5$

Explain clearly why, as usual, your teacher is wrong.

Solution. The blue and yellow triangles are in fact not similar. The yellow one has “(vertical) rise over (horizontal) run” of $3/8 = 0.375$ and the blue triangle has rise over run of $2/5 = 0.4$ so the

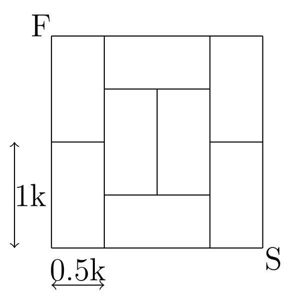
4 shapes as put together by your teacher looks like a triangle but is not! You can also see this as follows: The area of the yellow triangle is 12, the area of the green polygon is 8, the area of the blue triangle is 7 and the area of the blue polygon is 5. Hence they cover a total area of 32, which is neither 32.5 nor 31.5.

5. Friend or foe? A small function at Parliament House is attended by the following twelve people: Tony Abbott, Julie Bishop, Bob Brown, Julia Gillard, Joe Hockey, Barnaby Joyce, Rob Oakeshott, Tanya Plibersek, Christopher Pyne, Kevin Rudd, Wayne Swan and Malcolm Turnbull. These politicians, not being the best of friends, only shake hands with attendees whom they do not openly despise.
Tony shakes hands with one person, Julie with two, Bob with three, Julia with four, Joe with five, Barnaby with six, Rob with seven, Tanya with eight, Christopher with nine, Kevin with ten and Wayne with eleven.
With whom does Malcolm shake hands?

Solution. Wayne shakes hands with everyone, so he must shake hands with Malcolm and Tony. Therefore Tony *only* shakes hands with Wayne.
Kevin shakes hands with 10 people. Given that he does not shake Tony's hand, he shakes hands with everyone else, including Malcolm and Julie. Therefore Julie *only* shakes hands with Wayne and Kevin.
Christopher shakes hands with 9 people. Given that he does not shake hands with Tony and Julie, he shakes hands with everyone else, including Malcolm and Bob. Therefore Bob *only* shakes hands with Wayne, Kevin and Christopher.
Tanya shakes hands with 8 people. Given that she does not shake hands with Tony, Julie and Bob, she shakes hands with everyone else, including Malcolm and Julia. Therefore Julia *only* shakes hands with Wayne, Kevin, Christopher and Tanya.
Rob shakes hands with 7 people. Given that he does not shake hands with Tony, Julie, Bob and Christopher, he shakes hands with everyone else, including Malcolm and Joe. Therefore Joe *only* shakes hands with Wayne, Kevin, Christopher, Tanya and Rob.
Barnaby shakes hands with 6 people. Given that he does not shake hands with Tony, Julie, Bob, Christopher and Joe, he shakes hands with everyone else, including Malcolm.
Malcolm thus shakes hands with the following 6 people Barnaby, Rob, Tanya, Christopher, Kevin and Wayne. Interestingly, none of these is from his own party…

	Tony	Julie	Bob	Julia	Joe	Barnaby	Rob	Tanya	Christopher	Kevin	Wayne	Malcolm
Tony											☞	
Julie										☞	☞	
Bob									☞	☞	☞	
Julia								☞	☞	☞	☞	
Joe							☞	☞	☞	☞	☞	
Barnaby							☞	☞	☞	☞	☞	☞
Rob					☞	☞		☞	☞	☞	☞	☞
Tanya				☞	☞	☞	☞		☞	☞	☞	☞
Christopher			☞	☞	☞	☞	☞	☞		☞	☞	☞
Kevin		☞	☞	☞	☞	☞	☞	☞	☞		☞	☞
Wayne	☞	☞	☞	☞	☞	☞	☞	☞	☞	☞		☞
Malcolm						☞	☞	☞	☞	☞	☞	

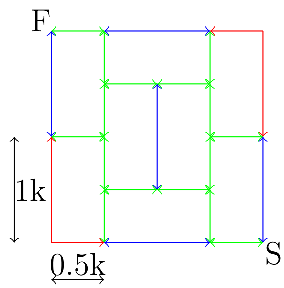
6. Fun run. The organisers of a fun run in the CBD have been given a number of streets by the Melbourne City Council which they can use for the run, as shown on the map below:



It is decided to start the race at S and finish at F, but a discussion breaks out among the organisers as to what the longest possible route from S to F is. All flunked their year 12 maths exam, and you are called in to settle the issue.
Find the length of the longest route from S to F (in kilometres) such that no section of street is used more than once, and show that no longer route exists, ending all discussion among the clueless organisers.
Each of the eight city blocks shown on the map are the same size: 0.5 by 1 kilometres.

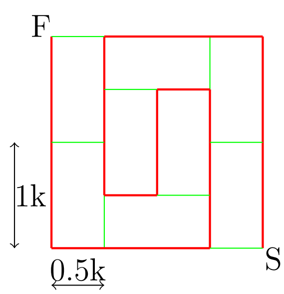
Solution. Call a point on the map where there is a choice of different directions an *intersection*. Apart from S and F, view an intersection as the meeting point of three *different* streets. Hence, even if you decide to go straight ahead at an intersection, you view this as entering a different street. The NE and SW corners are not intersections because at those corners there is no choice: you must turn a corner. Of course, at S and F two different streets meet. Using this kind of

counting of streets there are 22 streets 2 of which have length 1.5km (marked red) 5 have length 1km (marked blue) and 16 of which have length 0.5km (marked green).



The number of intersections is 14 plus the special intersections S and F, making a total of 16 intersections.

Each time you reach an intersection you have to choose between two options and once you have passed that intersection you can never return to it. This also applies to the start and finish; you have two directions in which to leave S and two in which to approach F, but you never return to S and you only reach F once. The upshot is that at each intersection one of the streets forming that intersection will have to remain unused for the fun run. Since there are 16 intersections, and at each intersection one street remains unused there are a minimum of 8 unused streets. If you make sure these are all of length 0.5km you must have found the longest possible route:



The longest possible run is 12km long.