These questions are designed to test your ability to analyse a problem and to express yourself clearly and accurately. The following suggestions are made for your guidance:

(1) Great weight will be attached by the examiners to the method of presentation of a solution. Candidates should state as clearly as they can the reasoning by which they arrived at their results. In addition, more credit will be given for an elegant than for a clumsy solution.

(2) The eight questions are not of equal length or difficulty. Generally, the later questions are more difficult than the earlier questions.

(3) It may be necessary to spend considerable time on a problem before any real progress is made.

(4) You may need to do considerable rough work but you should then write out your final solution neatly, stating your arguments carefully.

(5) Credit will be given for partial solutions; however a good answer to one question will normally gain you more credit than sketchy attempts at several questions.

Textbooks, electronic calculators and computers are NOT allowed. Otherwise normal examination conditions apply.
1. Find all positive integers \( L, M, N \) such that \( L^2 + M^2 = \sqrt{N^2 + 21} \).

2. Imagine a corridor lined with doors numbered 1, 2, etc (there is no upper limit). All the doors are closed. A controller then opens every second door. Then the controller reverses the state of every third door (that is to say, if it is open, the controller closes it, and if it is closed, the controller opens it). Then the controller reverses the state of every fourth door. Then every fifth door, and so on. What is the eventual state of the doors?

3. A merchant uses a set of scales to weigh his produce. Unfortunately, he dropped his 40kg weight, and it broke into four pieces. Out of interest, he had these weighed, and found that they each weighed a whole number of kilograms. Not only that, he realised, after some calculation, aided by his son who was a diligent student of mathematics, that these four pieces allowed him to weigh objects whose weight was any whole number of kilograms between 1 and 40. What were the weights of the four pieces? (Remember that he is using scales, so that he can put weights on both sides of the scale. For example, if he had an 8kg weight and a 10kg weight, he could put one on each side of his scales and thus weigh a 2kg object).

4. Simplify the following expression:

\[
\frac{(2^3 - 1)(3^3 - 1)\ldots (2011^3 - 1)}{(2^3 + 1)(3^3 + 1)\ldots (2011^3 + 1)}.
\]

5. There are 2011 jars labelled \( J_1, J_2, \ldots, J_{2011} \), which contain 2011\( n \) balls, distributed among the jars, for some positive integer \( n \). A legal move is to choose a label \( i \) between 1 and 2001, and move exactly \( i \) balls from jar \( J_i \) to any other particular jar. (That is to say, you can’t put the balls into several jars, only one). The aim is to use these moves to redistribute the balls so that there are exactly \( n \) balls in each jar, irrespective of the initial distribution of the balls. For what values of \( n \) is this possible?

6. Consider a regular pentagon with an inscribed star, as shown in the diagram. Is the area of the pentagon more than twice, less than twice or equal to twice the area of the star?

7. Find all solutions to the following system of equations:

\[
\begin{align*}
x + xy + xyz &= 12 \\
y + yz + yzx &= 21 \\
z + zx + zxy &= 30
\end{align*}
\]

8. Find all positive integer solutions of \((n - 1)! = n^k - 1\).