The University of Melbourne–Department of Mathematics and Statistics
School Mathematics Competition, 2012
(SENIOR DIVISION)

Time allowed: Three hours

These questions are designed to test your ability to analyse a problem and to express yourself clearly and accurately.

The following suggestions are made for your guidance.

1. Considerable weight will be attached by the examiners to the method of presentation of a solution. Candidates should state as clearly as they can the reasoning by which they arrived at their results. In addition, more credit will be given for an elegant than for a clumsy solution.

2. The seven questions are not of equal length or difficulty. Generally, the later questions are more difficult than the earlier questions.

3. It may be necessary to spend considerable time on a problem before any real progress is made.

4. You may need to do considerable rough work but you should then write out your final solution neatly, stating your arguments carefully.

5. Credit will be given for partial solutions; however a good answer to one question will normally gain you more credit than sketchy attempts at several questions.

Textbooks are NOT allowed. Electronic calculators, tables, etc., may be used. Computers may not be used. Calculators capable of storing text should have their memories erased before use. Otherwise normal examination conditions apply.

Candidates may attempt all questions.

Warning: Make sure you have the correct problems (Senior, Intermediate or Junior) in front of you.
1. Consider an $a \times b$ rectangle, as shown below, $a > b$. Clearly its area is $ab$. Now fold it on a diagonal, giving a five-sided figure, as in the diagram below. What is the area of the five-sided figure?

![Diagram of a rectangle and a five-sided figure](image)

2. In a river, there is an H-shaped bridge, with 5 arms labelled $a, b, c, d, e$. A sailing ship can’t sail upstream under the bridge, as its mast is too tall. An earthquake strikes, and each of the 5 arms of the bridge has, independently, probability $1/2$ of collapsing, thus allowing the ship to sail through the spot previously blocked by that arm. What is the probability that the ship can sail upstream after the earthquake? (You may assume that the earthquake did not damage the ship).

![Diagram of an H-shaped bridge](image)

3. Alan, Betty, Chris and Don play table tennis. Two of them play a game, and the loser leaves the table, to be replaced by one of the other two players who have sat out the game. The person who has sat out for the greatest number of consecutive games comes to the table for the next game. (If they have sat out for an equal number of games, either can come to the table). At the end of the day, Alan played 61 games, Betty played 22 games, Chris played 21 games and Don played 20 games. Who played in the 33rd game?

4. An odd number of people are standing in a field. The distance between each pair of people is distinct (i.e. different). They are each armed with a water pistol, and at the same precise moment each person fires at (and hits) the person nearest to them. Prove that at least one person does not get wet.
5. All vertices of a polygon $P$ lie at points with integer co-ordinates in the plane (that is to say, both their co-ordinates are integers), and all sides of $P$ have integer lengths. Prove that the perimeter of $P$ must be even.

6. It is asserted that one can find a subset $S$ of the nonnegative integers such that every nonnegative integer can be written uniquely in the form $x + 2y$ for $x, y \in S$. Prove or disprove the assertion.

7. Consider a game in which you have $n > 1$ identical coins. They are to be placed in a row. The first coin is placed down, and the second is then placed to the right of the first, the third is placed immediately to the right of the second, and so on. The rules of the game are that you can move a coin only when there is another coin immediately to the left of the coin you wish to move, and you can only place a coin down if there is a coin immediately to the left of the coin you are placing down. The aim of the game is to move a coin as far to the right as possible. With two coins you can’t move either of them. With three coins (as shown in the left-hand figure below), you can pick up the coin in position 2 and place it in position 4. You can’t move any coins further to the right. With $n > 3$ coins, what is the rightmost position to which you can move a coin, given a starting configuration in which the coins are placed in positions 1, 2, 3, . . . , $n$?

\[ \begin{array}{cccc}
\circ & \circ & \circ & \rightarrow \\
1 & 2 & 3 & 4 & \rightarrow \\
\end{array} \]

\[ \begin{array}{cccc}
\circ & \circ & \circ & \\
1 & 2 & 3 & 4 \\
\end{array} \]
1. Clearly, the area required is the area of the original rectangle less the area of the overlapping central triangle. The area of the original rectangle is \(ab\). Its diagonal is of length \(d = \sqrt{a^2 + b^2}\).

Let the height of the overlapping triangle be \(x\). Then its area is \(\frac{xd}{2}\).

Now \(\tan(\alpha) = \frac{b}{a} = \frac{2x}{d}\).

Therefore \(x = \frac{bd}{2a}\). Hence the area is

\[
ab - \frac{bd^2}{4a} = \frac{b}{4a} (3a^2 - b^2) .
\]

A check is to set \(a = b\) and note that the solution simplifies to \(ab/2\) as it must.

2. Each arm can, independently, be broken or unbroken. Thus there are \(2^5\) possible configurations. These can be enumerated, and it will be observed that a passage for the ship is available in 16 of the 32 cases, so that the required probability is \(1/2\). More elegantly, one can argue that for every open configuration there is a corresponding closed configuration, so the probability is \(1/2\). (This same argument will apply to more complicated bridge structures).

3. The total number of games played by the 4 players was 124, so 62 games were played (2 players per game). Alan played 61. If Alan did not play the first game, he played all the others, and won them all. Since Betty and Chris played more games than Don, they must have played the first game, won by Betty. Then the next games are played by Betty, Don, Chris in order, all of whom must lose, and that cycle of players continues, so that Don plays the 33rd game against Alan. If Alan plays the first game, he must win all games except the last, else he’d sit out two games. He must first play Betty, next Chris, then Don, and that cycle continues, until the last game (else they wouldn’t have played the stated number of games). Once again, this means Alan and Don played the 33rd game.

4. Consider the two closest people. They clearly shoot each other. The result then follows by induction. In a bit more detail, with only 3 players, the two closest shoot each other. The remainig
person stays dry. With 5 players, again the two closest shoot each other. If they are situated so that no-one shoots them, the remaining 3 players reduce to the previously solved problem. Otherwise, one of the remaining 3 shoots one of the two closest, hence one of the remaining 3 stays dry, as only two other shots are fired. On this basis an inductive proof can be built.

5. Assume that there are \( n \) vertices of the polygon. Label the vertices \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\), where we number the vertices sequentially around the polygon. The square of the length of the side joining vertex \( i \) to vertex \( i + 1 \) is just \((x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2\). So the sum of the squares of the lengths of all sides is

\[
\sum_{i=1}^{n} [(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2] = 2 \sum_{i=1}^{n} (x_i^2 - x_ix_{i+1}) + 2 \sum_{i=1}^{n} (y_i^2 - y_iy_{i+1}).
\]

Note that the last \((n^{th})\) vertex is connected to the first, so that \(x_{i+1} = x_1\) and \(y_{i+1} = y_1\). This is an even sum, and is the sum of the squares of the lengths. As the sum of the squares of the lengths is even, so is the sum of the lengths.

6. Yes one can. The subset \( S \) is the set of nonnegative integers that can be written with only 0s and 1s in base 4, so the first few elements are 0, 1, 4, 5, 16, 17, 20, 21, 64, .... To see this, let \( x \) be any nonnegative integer, and write \( x \) in base 4. Treating each digit separately, it is easy to write \( x = a + 2b \) with \( a, b \in S \), since 0 = 0 + 2 \times 0, 1 = 1 + 2 \times 0, 2 = 0 + 2 \times 1 \) and 3 = 1 + 2 \times 1. The expression is unique, for if \( x = a + 2b \) with \( a, b \in S \) then the digits of \( a \) and \( b \) are easy to determine, since \( a + b \) can be summed digit by digit independently. (Based on a solution of Matthew Ng).

7. This is actually a model in the mathematics physics literature due to Aldous and Diaconis. It is called the East model. Here is a solution due to Andrew Elvey Price:

First we may allow the moves of adding or removing a coin to or from a position with a coin to the left of it but restricting the number of coins to at most \( n \) without changing the answer. This is because the removed coins could instead just be placed at the left most empty position. We also note that all moves are reversible.

We will show by induction that the answer for \( k \geq 1 \) coins is \( 2^{k-1} \).

The base case is obvious. If there is one coin then there are no possible moves so the answer is 1. Suppose there are \( k + 1 \) coins. The following steps place a coin at position \( 2^k \)

- First remove the right most coin
- Use the rest to place one coin at point \( 2^{k-1} \)
- Put the removed coin at position \( 2^{k-1} + 1 \)
- reverse the moves used to place a coin at position \( 2^{k-1} \) (without moving the coin at position \( 2^{k-1} + 1 \)
• now do the algorithm for $k$ coins starting at position $2^{k-1} + 1$. This will place a coin at position $2^k$ as required.

Now suppose for sake of contradiction that it is possible to remove all but one (the one on the left) of the coins from the first $2^{k-1} + 1$ positions without removing all (but 1) of the coins.

Consider the leftmost coin in this distribution, excluding the one on the left. Let this coin be called $A$ and let it be at position $m$. Then $m > 2^{k-1} + 1$. Consider the last time when coin $A$ was placed here. At this point, there is a coin $B$ at point $m - 1 \geq 2^{k-1} + 1$. After this point, only $k - 1$ coins are used to remove $B$ from position $m - 1$. Reversing these steps places a coin at position $m - 1 > 2^{k-1}$ using only $k$ coins, contradicting the inductive assumption.

So there are always at least 2 coins in the first $2^{k-1} + 1$ positions (or off the table). So there are never more than $k - 1$ coins in positions other than the first $2^{k-1} + 1$. Looking at the moves in the rest of the positions (pretending that there is always a coin in position $2^{k-1} + 1$) give a legitimate sequence of moves using only $k$ coins with left most coin at position $2^{k-1} + 1$.

Hence it is impossible to get a coin past position $2^{k-1} + 2^{k-1} = 2^k$.

This completes the induction.

So the answer is $2^{n-1}$