1. The digits of 2013 are made up of consecutive numbers 0, 1, 2, 3 as will be true in the year 2453. What’s the most recent year prior to 2013 that this occurred?

Solution. The digits 20 are consecutive with 13 which occurs in 2013 and 2031 so the most recent year occurs before 2000. Any year in the 1000s with consecutive digits consists of 0123 or 1234. The largest number with these digits is 1432 so this is the most recent year.

2. Jim can make 6 origami roses in 7 minutes, Kim can make 5 origami roses in 6 minutes and Lim can make 4 origami roses in 5 minutes. Working at the same time, how long will it take them to make 225 origami roses for an upcoming party?

Solution. Jim, Kim and Lim work at similar rates so they will each make approximately $225/3 = 75$ origami roses. It takes Kim $15 \times 6 = 90$ minutes to make 75 roses. In 90 minutes Lim can make $18 \times 4 = 72$ roses and in 91 minutes Jim can make $13 \times 6 = 78$ roses. Hence in 91 minutes Jim, Kim and Lim make $78 + 75 + 72 = 225$ roses. (In that last minute, Kim and Lim may start to make roses but they will not complete them.)

3. Given a triangle of side lengths $a$, $b$ and 2, we draw one of its medians, dividing the side of length 2 into two equal parts. Given that the length of this median is also 2, find $a^2 + b^2$.

Solution. Drop a perpendicular with height $h$ which divides the base into $x + y = 1$.

Apply Pythagoras’ theorem:

$\begin{align*}
(1) \ h^2 + x^2 &= a^2, \\
(2) \ h^2 + y^2 &= 2^2, \\
(3) \ h^2 + (y + 1)^2 &= b^2
\end{align*}$

Hence

$(1) + (3) - 2(2) \Rightarrow x^2 + (y + 1)^2 - 2y^2 = a^2 + b^2 - 2 \times 2^2

= (1 - y)^2 + (y + 1)^2 - 2y^2 - 8

= 1 - 2y + y^2 + 1 + 2y + y^2 - 2y^2 + 8 = 10.$

The picture may not be correct since the perpendicular may lie outside the triangle or inside another triangle. Nevertheless, the proof is correct if we allow $x$ or $y$ to be negative (and still satisfy $x + y = 1$.)

4. For any real $r$, define $\lfloor r \rfloor$ to be the greatest integer less than or equal to $r$. So $\lfloor 4 \rfloor = 4$, $\lfloor \pi \rfloor = 3$, $\lfloor -1.3 \rfloor = -2$. Find all real $x$ that satisfy

$$\lfloor \frac{3}{x} \rfloor + \lfloor \frac{4}{x} \rfloor + \lfloor \frac{5}{x} \rfloor = 6.$$

Solution. Note that $x > 0$, since ($x \neq 0$ and) if $x < 0$ then each term on the left hand side is negative so cannot sum to 6. Furthermore, $x < 2$ since if $x \geq 2$ then the left hand side is $\leq 5$. So the equation is $1 + 2 + 3 = 6$ or $2 + 2 + 2 = 6$. In the first case

$$\lfloor \frac{3}{x} \rfloor = 1, \quad \lfloor \frac{4}{x} \rfloor = 2, \quad \lfloor \frac{5}{x} \rfloor = 3$$

(since $\lfloor \frac{3}{x} \rfloor \leq \lfloor \frac{4}{x} \rfloor \leq \lfloor \frac{5}{x} \rfloor$) so

$$\frac{3}{2} < x \leq \frac{3}{1}, \quad \frac{4}{3} < x \leq \frac{4}{2}, \quad \frac{5}{4} < x \leq \frac{5}{3} \quad \Rightarrow \quad \frac{3}{2} < x \leq \frac{5}{3}.$$

In the second case

$$\lfloor \frac{3}{x} \rfloor = 2, \quad \lfloor \frac{4}{x} \rfloor = 2, \quad \lfloor \frac{5}{x} \rfloor = 2$$

so

$$\frac{3}{3} < x \leq \frac{3}{2}, \quad \frac{4}{3} < x \leq \frac{4}{2}, \quad \frac{5}{3} < x \leq \frac{5}{2}$$

which has no solutions. Hence only the first case arises and $\frac{3}{2} < x \leq \frac{5}{3}$. 
5. Find all positive integers $a$, $b$ and $c$ that satisfy
\[ 7 \times a! + 13 \times b! = c! \]
where $n! = n \times (n - 1) \times \cdots \times 3 \times 2 \times 1$.

Solution. Note that $c$ is greater than $a$ and $b$ so $c! / a!$ and $c! / b!$ are integers.

Case 1. $a \leq b$
\[ 7 = \frac{b!}{a!} \times \left( \frac{c!}{b!} - 13 \right) = 7 \times 1 \text{ or } 1 \times 7. \]

If $b! / a! = 7$ and $c! / b! = 14$ then $b = 7, a = 6$ but then $c! / b! = 8, 72, \ldots \neq 14$.
If $b! / a! = 1$ and $c! / b! = 20$ then $c = 20, b = 19 = a$, or $c = 5$ and $b = 3 = a$.

Case 2. $a > b$
\[ 13 = \frac{a!}{b!} \times \left( \frac{c!}{a!} - 7 \right) = 13 \times 1 \]

so $a! / b! = 13$ and $c! / a! = 8$ thus $a = 13, b = 12$ and $c! / a! > 13$ and hence cannot equal 8.

Hence $(a, b, c) = (19, 19, 20)$ or $(3, 3, 5)$.

6. On each square of a $2013 \times 2013$ chessboard place either $+1$ or $-1$. Take the product, say
\[ (+1) \times (-1) \times (-1) \times \cdots = \pm 1, \]
along each row and add the products over all rows to get $R$. Similarly, take the product along each column and add the products over all columns to get $C$.

Is it possible to choose the $+1$s and $-1$s on the chessboard so that $R + C = 0$?

Solution. Consider an element $\epsilon = \pm 1$ in a row and a column with product $+1$ and $-1$. If we replace $\epsilon$ by $-\epsilon$ then $R + C$ is unchanged since the row and column products $+1$ and $-1$ switch to $-1$ and $+1$. If $\epsilon$ is in a row and a column both with product $+1$ then $\epsilon \mapsto -\epsilon$ causes $R + C \mapsto R + C - 4$ since both products change to $-1$ so $R \mapsto R - 2$ and $C \mapsto C - 2$. Similarly $R + C \mapsto R + C + 4$ if an element in a row and a column both with product $-1$ is flipped. Any state, i.e. assignment of $\pm 1$ on the chessboard, can be obtained from any other by flipping values. Hence we have proven that $R + C$ is the same mod 4 for any two states. If we place $+1$ on each square then $R + C = 4026 = 2 \text{(mod 4)}$. Thus any state satisfies $R + C = 2 \text{(mod 4)}$ so in particular $R + C$ can never be zero.

(Instead of placing $+1$ on each square, it is easy to see that given a state of a $k \times k$ chessboard we can always find a state of a $(k + 2) \times (k + 2)$ chessboard with the same $R + C$. The $1 \times 1$ chessboard containing 1 has $R + C = 2$ so we can find a state of the $2013 \times 2013$ chessboard with $R + C = 2$. Thus any state satisfies $R + C = 2 \text{(mod 4)}$.)

7. Is it possible to have a tournament of 9 players so that each game consists of 3 players competing as individuals against each other, such that each player plays every other player exactly once?

Solution. Each player should play 4 games in order to meet the other 8 players. Hence, there should be $12 = 4 \times 9 / 3$ games. The first games are:
\[ (1, 2, 3), \ (1, 4, 5), \ (1, 6, 7), \ (1, 8, 9), \ (2, 4, 6), \ (2, 5, 8), \ (2, 7, 9). \]

This is general, i.e. we can relabel the players so that these 7 games occur. Note that $(2, 5, 7)$ is ruled out because it would leave $(2, 8, 9)$ which violates $(1, 8, 9)$.

It remains to find 5 more games. Without loss of generality, the next 3 games involve $3$. There are three possibilities: $(3, 4, 7), \ (3, 5, 9), \ (3, 6, 8)$ or $(3, 4, 9), \ (3, 5, 6), \ (3, 7, 8)$ or $(3, 4, 9), \ (3, 5, 7), \ (3, 6, 8).$ The first two lead to violations: $(4, 8, 9)$ in the first case and $(4, 7, 8)$ in the second. The third allows the two final games $(4, 7, 8)$ and $(5, 6, 9)$. Hence the tournament is possible, with games:
\[ (1, 2, 3), \ (1, 4, 5), \ (1, 6, 7), \ (1, 8, 9), \ (2, 4, 6), \ (2, 5, 8), \ (2, 7, 9), \ (3, 4, 9), \ (3, 5, 7), \ (3, 6, 8), \ (4, 7, 8), \ (5, 6, 9). \]