1. By construction, the length $AE$ is equal to the length $A'E$, the length $AD$ is equal to the length $A'D$, so the sum of the lengths of the sides of the shaded triangle is the sum of the lengths of the original triangle, which is 3 units. Thus each shaded triangle has perimeter 1 unit.

2. Let the rectangle be of dimension $a \times b$. Thus the area $A$ is $ab$ and the perimeter $P$ is $2(a + b)$. Solving these equations leads to a quadratic equation,

$$2b^2 - Pb + 2A = 0,$$

which has two solutions, $a$ and $b$, which are the two side-lengths. In the degenerate case when $a = b$, the rectangle is a square. So the answer is yes.

3. One can factorize $m^n - 1$ as $(m - 1)(1 + m + m^2 + \cdots + m^{n-1})$. So if $m^n - 1$ is to be a prime, one of the factors must equal 1. Thus $m = 2$. This is the only possibility, as the second factor cannot equal 1.

4. No. Label the positive integers, in sequence, $m_1, m_2, \ldots, m_{2013}$. Then we must have $\frac{m_i}{m_{i+1}} = p_i^{\pm 1}$ for $i \in [1, 2012]$, where $p_i$ denotes a prime. Furthermore, $\frac{m_1}{m_{2013}} = p_{2013}^{\pm 1}$. Now

$$\frac{m_1}{m_{2013}} = \frac{m_1}{m_2} \frac{m_2}{m_3} \cdots \frac{m_{2012}}{m_{2013}}.$$ 

So

$$p_{2013} = \frac{\text{A product of } k \text{ primes}}{\text{A product of } (2012 - k) \text{ primes}}.$$ 

If $k$ is even, so is $2012 - k$, and similarly if $k$ is odd, so must $2012 - k$ be. Cross multiplying, we have that the product of an odd number of primes is equal to the product of an even number of primes, which cannot be.

5. First, break the square into 4 smaller squares, each of unit area. Given that there are 9 points, there must be at least one square containing at least three points. So we
seek the maximum area of a triangle inscribed in a unit square. Clearly, if the three points occupy three vertices of a square, and using the fact that the area of a triangle is $1/2$ (base $\times$ height), the area of the triangle is 1/2 square units. We claim that this is the (non-unique) maximal area of a triangle inscribed in a square. To prove this, we first show that one vertex lies at the corner of the square. To see this, choose any edge of the triangle and consider the vertex opposite that edge. We can increase the area of the triangle by moving the vertex away from the opposite edge until it hits the side of the square, and can then further increase the area by moving the vertex to the corner of the square that maximises the height of the triangle.

Now repeat this argument/construction with the other two vertices. Moving them toward the appropriate vertices of the square will maximise the area of the triangle. Thus we end up with a triangle with vertices at the corner of the square. (A simpler, more elegant proof can be constructed by a folding argument). This has area $1/2$.

6. For the product of all the numbers thrown to be divisible by 14, this product must include factors of both 2 and 7. Four numbers in the range 1, $\cdots$, 9 are divisible by 2, and one is divisible by 7. Hence the probability that there is no factor of 2 or 7 after $n$ throws of the dice is $(\frac{4}{9})^n$. The probability that there is no 7 is $(\frac{8}{9})^n$, so the probability that there is a 2 but no 7 is the difference of these two probabilities. Similarly, the probability that there is a 7 but no 2 is $(\frac{5}{9})^n - (\frac{4}{9})^n$. So the only remaining outcome is getting both a 2 and a 7, thus making the probability divisible by 14. That is,

$$1 - \left(\frac{4}{9}\right)^n - \left(\frac{8}{9}\right)^n + \left(\frac{4}{9}\right)^n - \left(\frac{5}{9}\right)^n + \left(\frac{4}{9}\right)^n = 1 - \left(\frac{8}{9}\right)^n - \left(\frac{5}{9}\right)^n + \left(\frac{4}{9}\right)^n.$$ 

7. (Solution by Andrew Elvey-Price). Claim: There exists such a sequence.
For positive integers $k$ and $n$, let $g_k(n)$ be the number of interesting numbers in the sequence $(n+1, n+2, \ldots, n+k)$. Then it remains to prove that $g_{2013}(n) = 750$ for some $n$
We will prove this via a series of lemmas.

**Lemma 1:** $g_{2013}(0) \geq 750$
**Lemma 2:** There is some $c$ such that $g_{2013}(c) = 0$
**Lemma 3:** For each positive integer $n$, we have $g_{2013}(n + 1) \geq g_{2013}(n) - 1$

Then as a result of these lemmas, $g(0), g(1), \ldots, g(c)$ is a sequence of integers with $g(0) \geq 750$ and $g(c) < 750$.

Therefore, there must be some integer $m$, between 1 and $c$ such that $g_{2013}(m) \geq 750$ and $g_{2013}(m + 1) < 750$. But $g_{2013}(m + 1) \geq g_{2013}(m) - 1$.

Therefore, $751 > g_{2013}(m + 1) + 1 \geq g_{2013}(m) \geq 750$. But $g_{2013}(m)$ is an integer.
Therefore, $g_{2013}(m) = 750$, as required.

**Proof of Lemma 1:**
If a number \( m = ab \) where \( a \) and \( b \) are both less than or equal to 300, then every prime factor \( p \) of \( m \) is a factor of \( a \) or \( b \), so \( p \leq 300 \). So \( m \) is interesting.

Therefore, the 300 numbers 1, 2, \ldots, 300 are interesting.
The 150 numbers \( 2 \times 151 = 302, 2 \times 152, \ldots, 2 \times 300 = 600 \) are interesting.
The 100 numbers \( 3 \times 201 = 603, 3 \times 202, \ldots, 3 \times 300 = 900 \) are interesting.
The 75 numbers \( 4 \times 226 = 904, 4 \times 227, \ldots, 4 \times 300 = 1200 \) are interesting.
The 60 numbers \( 5 \times 241 = 1205, 5 \times 242, \ldots, 5 \times 300 = 1500 \) are interesting.
The 50 numbers \( 6 \times 251 = 1506, 6 \times 252, \ldots, 6 \times 300 = 1800 \) are interesting.
The 30 numbers \( 7 \times 258 = 1806, 7 \times 259, \ldots, 7 \times 287 = 2009 \) are interesting.
Since these numbers are all different, there are at least
\[
300 + 150 + 100 + 75 + 60 + 50 + 30 = 765 > 750
\]
interesting numbers in the sequence \((1, \ldots, 2013)\).
Therefore, \( g_{2013}(0) > 750 \). This completes the proof of lemma 1.

**Proof of Lemma 2:**
Suppose for sake of contradiction that \( 2013^2 + k \) is interesting for some \( k \in \{1, 2, \ldots, 2013\} \).
Then clearly \( k|2013! \). Now let \( p \) be a prime factor of \( \frac{2013^2}{k} + 1 \). Then \( p|2013^2 + k \) so \( p < 300 \). Therefore, \( p|2013! \frac{2013^2}{k} \). So \( p|\frac{2013^2}{k} + 1 - \frac{2013^2}{k} = 1 \). A contradiction.
Therefore \( 2013^2 + k \) is not interesting for any \( k \in \{1, 2, \ldots, 2013\} \).
So none of the numbers \( 2013^2 + 1, 2013^2 + 2, \ldots, 2013^2 + 2013 \) are interesting.
So \( g_{2013}(2013^2) = 0 \). This completes the proof of lemma 2.

**Proof of lemma 3:**
If \( n \) is interesting then \( g_{2014}(n) = 1 + g_{2013}(n + 1) \)
If \( n \) is not interesting, then \( g_{2014}(n) = g_{2013}(n + 1) \)
So \( g_{2013}(n + 1) \geq g_{2014}(n) - 1 \). Also, \( g_{2014}(n) \geq g_{2013}(n) \)
Therefore, \( g_{2013}(n + 1) \geq g_{2013}(n) - 1 \). This completes the proof of lemma 3.