The University of Melbourne–Department of Mathematics and Statistics
School Mathematics Competition, 2016
INTERMEDIATE DIVISION: SOLUTIONS

(1) In the following sum substitute each letter for a different digit from the set \{1, ..., 9\}.

\[
\begin{array}{c}
BAD \\
+BOB \\
SOBS
\end{array}
\]

Solution. Since BAD and BOB are less than 999 the sum is less than 2000 hence \(S = 1\).

Since \(S = 1\), we must have \(B = 5, 6, 7, 8\) or \(9\). If \(B = 5\) then \(O = 0\) or \(1\) which is impossible so \(B \neq 5\).

If \(B = 6\) then \(O = 2\) or \(3\) hence \(O = 2\) since \(O + A + 1 < 16\), so \(A = 3\) and \(D = 5\) thus we have

\[
\begin{array}{c}
635 \\
+626 \\
1261
\end{array}
\]

If \(B = 7\) then \(O = 4\) or \(5\) hence \(O = 4\) since \(O + A + 1 < 17\), so \(D = 4 = O\) which is impossible.

If \(B = 8\) then \(O = 6\) or \(7\) hence \(O = 6\) since \(O + A + 1 < 18\), so \(A = 1 = S\) which is impossible.

If \(B = 9\) then \(O = 8\), so \(A = 0\) which is impossible. Hence we have exactly one solution.

(2) Fill in the diagram on the left with nine numbers so that adjacent numbers sum to the number outside the polygon. For example, the diagram on the right shows the numbers 7, 8 and 5 filled in.

\[
\begin{array}{c}
57 \\
66 \\
63 \\
64 \\
71 \\
59 \\
61 \\
69 \\
60
\end{array}
\]

\[
\begin{array}{c}
15 \\
8 \\
5 \\
7
\end{array}
\]

Solution. Twice the interior number between 57 and 63 is given by the alternating sum

\[
57 - 66 + 69 - 60 + 61 - 59 + 71 - 64 + 63 = -9 + 9 + 2 + 7 + 63 = 72
\]
so place $36 = 72/2$ in that position and fill in the remaining numbers.

(3) Four classmates hang up their identical hats. Later they all randomly pick a hat. What is the probability that exactly one of the classmates has his own hat?

**Solution.** There are $4! = 24$ different ways to put on hats. Of these, the number of ways of having exactly $k$ classmates with the correct hat for $k = 4, 3, 2, 1, 0$ is $1, 0, 6 = \binom{4}{2}, 8 = 2 \times \binom{4}{1}, 9$. Note that $1 + 0 + 6 + 8 + 9 = 24$. Hence the probability that exactly one of the classmates has his own hat is $8/24 = 1/3$.

(4) Beginning with a natural number $\{0, 1, 2, 3...\}$, take the sum of its digits to get a new natural number, then take the sum of its digits and so on until you reach a single digit natural number. Write $\oplus(n)$ for this function applied to the natural number $n$. For example

$$77 \mapsto 7 + 7 = 14 \mapsto 1 + 4 = 5 \Rightarrow \oplus(77) = 5.$$ 

Similarly, define $\otimes(n)$ by taking products of digits in place of sums, for example

$$77 \mapsto 7 \times 7 = 49 \mapsto 4 \times 9 = 36 \mapsto 3 \times 6 = 18 \mapsto 1 \times 8 = 8 \Rightarrow \otimes(77) = 8.$$ 

Calculate $\oplus(5^{2016})$ and $\otimes(5^{2016})$.

**Solution.** $\oplus(5^{2016}) = 1$

Notice that $\oplus(n)$ = the remainder of $n$ after division by 9, or 9 if $n$ is divisible by 9. This follows by repeatedly applying that fact that the sum of the digits of a number $k$ is equal to $k$ mod 9 since $10^m \equiv 1 \mod 9$ for $m = 0, 1, 2, \ldots$. Now $5^3 = 125 \equiv -1 \mod 9$ hence $5^{2016} = (5^3)^{672} \equiv (-1)^{672} \equiv 1 \mod 9$ so $\oplus(5^{2016}) = 1$.

$\otimes(5^{2016}) = 0$

Notice that $5^3 = 125$, $5^4 = 625$ and in fact for $n > 1$ the last 2 digits of $5^n$ is always given by 25 since $5 \times 25 = 125$. Hence the last two digits of $5^{2016}$ is given by 25. Hence, the product of the digits of $5^{2016}$ includes the factor $2 \times 5 = 10$ and the product of the digits of this number is 0, i.e. $\otimes(5^{2016}) = 0$.

(5) The years 2016 and 2000 share the property that the sum of their prime factors is small:

$$2016 = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 7 \rightarrow 2 + 2 + 2 + 2 + 3 + 3 + 7 = 23$$

$$2000 = 2 \times 2 \times 2 \times 2 \times 5 \times 5 \times 5 \rightarrow 2 + 2 + 2 + 2 + 5 + 5 + 5 = 23.$$ 

How soon will we see a year such that the sum of its prime factors is $< 23$?

**Solution.** Notice that

$$2048 = 2^{11}, \quad 11 \times 2 = 22 < 23$$
so either 2048 is the next year such that the sum of its prime factors is $< 23$ or the year is one of the 31 years

$$2017, 2018, \ldots, 2047.$$  

Each factor must be less than 23. In fact each factor must be less than 7 by the following argument which we will apply to 7 (but it equally applies to 11, 13, 17, 19). If 7 (or 11, 13, 17, 19) is a factor, then we will find that the maximum product, under the constraint that the sum of the prime factors is $< 23$, is less than 2017 hence too small. We prove this claim using a strategy that maximises products keeping the sum the same. We may assume there is exactly one factor of 7 (or 11, 13, 17, 19) using the same maximising strategy below. The strategy:

- replace $2^3$ by $3^2$
- replace 5 by $2 \times 3$

This strategy removes 5s so for $2a + 3b + 5c < 16 = 23 - 7$ we have $2^a 3^b 5^c 7 \leq 2^4 3^B 7 \leq 3^5 7 = 1781$ for $2A + 3B < 16$ where $A = 0, 1$ or 2. Since 1781 $< 2017$ there cannot be a factor of 7 (nor 11, 13, 17, 19 by the same argument).

Hence the only factors are 2, 3, 5 so the year factorises as:

$$2^a 3^b 5^c, \quad 2a + 3b + 5c < 23.$$  

If both 2 and 5 are factors then the year has a factor of 10 hence is 2020, 2030 or 2040. But 202, 203, 204 have prime factors greater than 2, 3 and 5 so this does not occur.

The nearby powers of 2 are 512, 1024, 2048 and the nearby powers of 5 are 625, 3125 so the year cannot be a power of 2 (unless it is 2048) or a power of 5, and furthermore it cannot have a single factor of 3 since 3 times the powers above—1536, 3072 and 1875—do not lie between 2017 and 2047. Hence there is at least a factor of 9, which is satisfied by 2025, 2034 and 2043. We have

$$2025 = 3^4 5^2 = 3 \times 3 \times 3 \times 3 \times 5 \times 5, \quad 3 + 3 + 3 + 3 + 5 + 5 = 22 < 23$$

hence 2025 is the next year when the sum of its prime factors is $< 23$.

(6) Each vertex of a right-angle triangle is reflected in the opposite side. Prove that the area of the triangle determined by the three points of reflection $P$, $Q$, $R$ in the diagram, is equal to three times the area of the original triangle.

![Diagram of a right-angle triangle with vertices P, Q, R and their reflections]
Solution.

In the diagram the perpendicular from $P$ to the line $QR$ is perpendicular to the line $BC$ and goes through $A$. Hence the height of the triangle $PQR$ is 3 times the height of the triangle $ABC$ and with common base of length the length of $BC$. Thus the area of $PQR$ is 3 times the area of $ABC$.

A second solution uses a tiling of the plane by copies of $PQR$ or copies of $ABC$ and it is easy to see that 2 copies of $PQR$ covers the same area as 6 copies of $ABC$. 


(7) In Sybille’s mathematics class, the following game is played: the teacher writes on the board three numbers strictly between 0 and 2016. The three numbers are not all the same. At each turn, a student erases all of the numbers on the board and replaces them with the averages of each pair of numbers. So if \((a, b, c)\) are on the board the student replaces them with

\[
(a, b, c) \mapsto \left(\frac{a + b}{2}, \frac{a + c}{2}, \frac{b + c}{2}\right).
\]

After 10 students finish their turns, there are three integers on the board, the smallest of which is 500. What is the largest number on the board at that point?

**Solution.** The difference between the smallest and largest of the three numbers is halved at each step. After 10 steps the difference between the smallest and largest of the three numbers is less than \(2016/1024 < 2\). If the final difference between the smallest and largest of the three numbers is 0 then the three numbers are equal and hence equal at every step. But the initial three numbers are not all the same. Hence the final difference between the smallest and largest of the three numbers is 1 and the largest number on the board is 501.

A second proof. By going backwards

\[(a, b, c) \mapsto (a + b - c, a - b + c, -a + b + c)\]

we see that if the final three numbers are integers then the three numbers are integers at any step. To maintain integers at every step, one can start with \((d, d, d)\) for any integer (which of course did not occur in Sybille’s mathematics class). Any other solution—integers at every step—must differ from \((d, d, d)\) by a factor of 1024 so that it can be halved 10 times. Hence the beginning three numbers must be

\[(1024, 0, 0) + (d, d, d) \quad \text{or} \quad (1024, 1024, 0) + (d, d, d)\]

so the final three integers are

\[(1, 0, 0) + (d, d, d) \quad \text{or} \quad (1, 1, 0) + (d, d, d)\]

Since the smallest of the final three integers is 500 then \(d = 500\) and the largest number on the board is 501.