The University of Melbourne–Department of Mathematics and Statistics
School Mathematics Competition, 2016

JUNIOR DIVISION

Time allowed: Two hours

These questions are designed to test your ability to analyse a problem and to express yourself clearly and accurately. The following suggestions are made for your guidance:

(1) *Considerable weight will be attached by the examiners to the method of presentation of a solution. Candidates should state as clearly as they can the reasoning by which they arrived at their results. In addition, more credit will be given for an elegant than for a clumsy solution.*

(2) *The **six** questions are not of equal length or difficulty. Generally, the later questions are more difficult than the earlier questions.*

(3) *It may be necessary to spend considerable time on a problem before any real progress is made.*

(4) *You may need to do considerable rough work but you should then write out your final solution neatly, stating your arguments carefully.*

(5) *Credit will be given for partial solutions; however a good answer to one question will normally gain you more credit than sketchy attempts at several questions.*

*Textbooks, electronic calculators and computers are **NOT** allowed. Otherwise normal examination conditions apply.*

1. Queen Mab. The wicked Queen Mab is mixing a magic potion in order to put a spell on Lancelot. She has two large cauldrons, one containing 9 litres of dragon blood and the other containing 6 litres of unicorn blood. She pours 3 litres of the dragon blood into the cauldron containing the unicorn blood. She thoroughly stirs, and then pours 3 litres of the mixture back into the cauldron with dragon blood. How much dragon blood is in each of the cauldrons, assuming Queen Mab has not spilled a drop of blood?

Solution: After pouring 3 litres of blood from the cauldron with dragon blood into the cauldron with unicorn blood, the former contains 6 litres of dragon blood and the latter 3 litres of dragon blood and 6 litres of unicorn blood, giving a total of 9 litres of mixed blood. If we now pour back 3 litres of the mixture, this amounts to one third of all of the mixture, and hence also one third of the 3 litres of dragon blood it contains. We thus end up with one cauldron containing 7 litres of dragon blood (and 2 litres of unicorn blood), and one cauldron containing 2 litres of dragon blood (and 4 litres of unicorn blood).

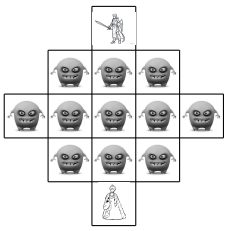
2. The Panama papers. On April 3 the so-called "Panama papers" were leaked, detailing the shady tax affairs of many very wealthy people, including prime ministers, arms dealers and celebrities. A man who called himself "John Doe" stole the papers, contained in three large boxes, from the firm Mossack Fonseca in Panama City. Together with his trusted helper, "Jane Roe", he brought them to safety in neighbouring Costa Rica. To avoid being discovered crossing the border, John and Jane used a 20 km hidden track through the mountains. Because the track was very steep, they could only carry one box each at a time and walk at a constant pace of 1 km per hour (whether carrying a box or not). What is the shortest time it could have taken John and Jane to carry the three boxes from the start to the finish of the track?

Solution: The problem is that there are three boxes and only two people. If John and Jane walk together across the track, each carrying one box, then one of them would have to go back to get the last remaining box while the other person has to wait for 40 hours. In this scenario they take 60 hours total, one of them doing 20 hours of walking and 40 hours of waiting while the other does 60 hours of walking.
They can do better, however, so that not a single minute is lost by one person waiting. The solution is for John and Jane to both carry a box for 10 km (the half-way mark). Then Jane drops her box and returns to pick up the third box while John continues walking towards the finish. After 20 hours total John has carried his box across, Jane is back at the start of the track and one box is stranded at the half-way point. John now walks back and Jane starts walking with the third box. After another 10 hours they meet up again at the half-way mark. John picks up the dropped box and together they complete the last 10 km, taking another 10 hours. In total is has now taken them 40 hours. Since at all times they are actively contributing to getting the boxes from start to finish (rather than one person being idle) they cannot possibly be more efficient than this.

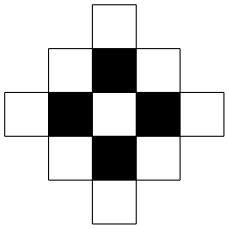
3. The Donald. US presidential hopeful Donald Trump's most prized possession is his toupée. There is always a lot of hot air blowing around Mr Trump, so for his upcoming tour around the United States he has decided to bring plenty of spare hairpieces, taking with him n containers, each containing 1000 toupées. Each container is carried on board his private jet by exactly 7 of his personal assistants (PAs), and each of his PAs helps carry exactly 10 containers. If Mr Trump has at least 65 and at most 70 PAs, how many toupées is he taking with him on his tour?

Solution: If every PA were to carry only 1 container and each container is carried by 7 PAs this would mean that Mr Trump needs $7 \times n$ PAs to carry the n containers. However, each PA carries 10 containers in total, so that he only needs $7 \times n/10$ PAs. Since 7 and 10 do not have a common factor, this means that n must be a multiple of 10 and, more importantly, the number of PAs must be a multiple of 7. The only multiple of 7 between 65 and 70 is 70. Hence Mr Trump has 70 PAs and $7 \times n/10 = 70$ so that $n = 100$. Mr Trump thus takes $1000 \times 100 = 100,000$ toupées with him on his trip. In fact, if he is also wearing one (which most likely he is) it would be $100,001$ toupées.

4. A knight's tale. To reach his beloved Guinevere, Lancelot must defeat 11 hideous monsters. Unfortunately, the evil Queen Mab has put a spell on Lancelot, and he can only make knight-moves (two down and one right/left, one down and two right/left, one up and two right/left, or two up and one right/left). Can he defeat all 11 monsters and reach Guinevere without occupying each square more than once (and without leaving the kingdom, consisting of the 13 squares)? If your answer is yes, show a way Lancelot can reach Guinevere. If your answer is no, explain clearly why Lancelot must fail.



Solution: Colour the squares of the kingdom in black and white as in a chess board as shown on the right. Each knight move takes Lancelot from a white to a black or from a black to a white square. This shows that he can never reach the white square in the centre, so that Lancelot and Guinevere are doomed by Queen Mab's wicked spell.



5. The Rio Olympics. At the upcoming Rio Olympic Games there will be two new sports: rugby sevens and golf. In golf, countries will compete against each other in teams of 4 players. Each player in a team plays 4 holes, and the total score of a team is the sum of the score of the individual players. For example, if the players take 8, 6, 4 and 5 shots respectively to complete their 4 holes, then the team's score is $8 + 6 + 4 + 5 = 23$. Recently, Australia played an Olympic qualifier against New Zealand. Australia's total score was 30, and New Zealand's was 32. Of the eight players, no two achieved the same score. The best (= lowest) scoring player was from New Zealand while the worst (= highest) scoring player was from Australia. What were the scores of the 4 Australian players?

Solution: The lowest possible score for a player is 4, requiring them to score a hole-in-one for each of their 4 holes. Since all 8 players have a different score, the lowest score of the two teams combined is

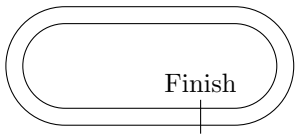
$$4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 = (4 + 11) + (5 + 10) + (6 + 9) + (7 + 8) = 4 \times 15 = 60.$$

We know that the total score was $30 + 32 = 62$, and the only possible ways to achieve the extra 2 is as

$$\text{(A)}: \ 4 + 5 + 6 + 7 + 8 + 9 + 10 + 13 \quad \text{or} \quad \text{(B)}: \ 4 + 5 + 6 + 7 + 8 + 9 + 11 + 12.$$

First consider option (A). Then NZ has a 4 and Australia a 13. But Australia needs a total score which is two better than that of NZ, so they need to make up $9 + 2 = 11$ points. Even if they scored the three next-best scores: 5, 6, 7 (leaving NZ with 8, 9, 10) they would only make up $3 + 3 + 3 = 9$ points, drawing level with NZ. We can thus dismiss option (A).
Next consider option (B). NZ again has a 4 and Australia now has a worst score of only 12, needing to make up 10. This time this is possible in exactly one way: $5 + 6 + 7$ is 10 less than $8 + 9 + 11$. Hence the Australian scores were 5, 6, 7, 12 and those of NZ were 4, 8, 9, 11.

6. The Melbourne Cup. Melbourne's Flemington racecourse is 2300 metres long. Horses Prince Of Penzance, Protectionist, Fiorente, Green Moon, Dunaden, Americain, Shocking, Viewed, Efficient, Delta Blues and Makybe Diva are each placed at a random but different position along the race track. At the sound of the starter's pistol, all horses start running at a constant speed of 72 kilometres per hour. Depending on the direction in which a horse is facing, it runs towards the finish line in either a clockwise or anti-clockwise direction. There is a twist in the race, however. Whenever two horses meet along the course, they both turn around and start running towards the finish line in the opposite direction (still at the same speed of 72 kilometres per hour). How long must you wait before you know with 100% certainty that all horses have crossed the finish line? Give your answer in seconds.



Solution: Two horses meeting along the course and reversing direction does not at all affect the answer to this question. Yes, they swap directions, but there will still be one horse running clockwise and one running anti-clockwise at a speed of 72 km/hour = 20 metres/second. Now the worst that can happen is that a horse is placed exactly on the finish line at the start of the race. It then takes exactly the time required to complete a full lap for every horse to have finished. At a speed of 20 metres/second it takes $2300/20 = 115$ seconds to complete a full lap of the Flemington racecourse.
The answer is thus 115 seconds.