1. In the sequence of positive integers, 2016 is the 2016th number. If we delete all squares and cubes from the sequence, in which position does 2016 now sit?

As $44^2 = 1936 < 2016 < 45^2 = 2025$ there are 44 squares less than or equal to 2016. As $12^3 = 1728 < 2016 < 13^3 = 2197$ there are 12 cubes less than or equal to 2016. However, the sixth powers $1^6 = 1^3 = 1^2 = 1$, $2^6 = 4^3 = 8^2 = 64$ and $3^6 = 9^3 = 27^2 = 729$ have been double counted, hence the number of numbers remaining is $2016 - 44 - 12 + 3 = 1963$.

2. Take any prime number greater than 3. Square it, then add 14. Divide your answer by 12. Prove that the remainder is always 3.

Any prime greater than 3 can be written as $6x + 1$ or $6x - 1$.

Now $(6x + 1)^2 + 14 = 36x^2 + 12x + 15 = 12(3x^2 + x + 1) + 3$.

so always has remainder 3 when divided by 12. Similarly, $(6x - 1)^2 = 12(3x^2 - x + 1) + 3$.

3. Is $10^{100} + 1$ a prime number?

If $m$ is odd, then $x^m + 1 = (x+1)(x^{m-1} - x^{m-2} + \ldots - x + 1)$. Therefore, $10^{100} + 1 = (10^4)^{25} + 1$ is divisible by $10^4 + 1$ and hence is not a prime.

4. Consider the number 3025. The two-digit integer 55 has the property that the square of the sum of the first two digits (30) and the last two digits (25) is $55^2 = 3025$, the original number. Find another four-digit number with this property.

There are only two other such four-digit numbers: 2025 and 9801.

Let $a$ be the number formed by the first two digits of a four-digit number with the desired property and $b$ be the number formed by the last two digits of the same four-digit number. Note that $10 \leq a \leq 99$ and $0 \leq b \leq 99$.

We have that

$$(a + b)^2 = 100a + b,$$

where $100a + b$ is the four-digit number we seek.

Consider the remainder of both sides when divided by 99. These must be the same as the dividends are equal. As $99a$ is always divisible by 99, $100a + b - 99a = a + b$ must have the same remainder as $100a + b$ when divided by 99.

Writing $K = a + b$, $K$ and $K^2$ have the same remainder when divided by 99. Moreover, $K^2$ is a four-digit number, so $K$ is an integer between 32 and 99 (inclusive; $31^2 = 961 < 1000 \leq 32^2 = 1024$ and $99^2 = 9801 \leq 9999 < 100^2 = 10000$).

Immediately we can try $K = 99$. This yields $K^2 = 9801 = (98 + 1)^2$, so 9801 satisfies the desired property.
For the other solutions, as $K$ and $K^2$ have the same remainder when divided by 99, their difference $K^2 - K = K(K - 1)$ must be divisible by 99.

The prime factorisation of 99 is $3^2 \cdot 11$. As $K$ and $K - 1$ share no prime factors, we must have that 11 divides one and $3^2 = 9$ divides one (possibly the same). As $32 \leq K \leq 99$, this is only possible when $K \in \{45, 55, 99\}$. $K = 55$ will yield 3025 as the four-digit number, while $K = 45$ will yield a new solution, $2025 = (20 + 25)^2$, as the four-digit number.

5. Show that, for any triangle, the area $A$ and perimeter $P$ satisfy the inequality $A \leq \frac{P^2}{12\sqrt{3}}$.

Let the three sides of the triangle be $a$, $b$, and $c$. Let

$$s = \frac{a + b + c}{2}.$$ 

The perimeter $P = 2s$, and the area is given by Heron’s formula:

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$ 

The arithmetic mean–geometric mean inequality tells us that

$$(x \cdot y \cdot z)^{1/3} \leq \frac{x + y + z}{3}.$$ 

An immediate application of this inequality gives the result

$$[(s-a)(s-b)(s-c)]^{1/3} \leq \frac{s-a+s-b+s-c}{3} = \frac{3s-2s}{3} = \frac{P}{6},$$

so

$$A = \sqrt{s \left( [(s-a)(s-b)(s-c)]^{1/3} \right)^{3/2}} \leq \sqrt{\frac{P}{\sqrt{2}}} \left( \frac{P}{6} \right)^{3/2} = \frac{P^2}{12\sqrt{3}}.$$ 

6. Let $a$, $b$, $c$ be integers. Prove that

$$abc(a^3 - b^3)(b^3 - c^3)(c^3 - a^3)$$

is always divisible by 7.

If any of $a, b, c$ are 0 mod 7 or equal mod 7 the problem is done. So they must be 1, 2, 3, 4, 5, or 6 mod 7, and unequal. Note that $a^3 \text{ mod } 7$ is congruent to 1 if $a = 1, 2, 4,$ and 6 if $a = 3, 5, 6$. Therefore by the Pigeonhole Principle at least two of $a^3, b^3$ and $c^3$ must be congruent mod 7. So $abc(a^3 - b^3)(b^3 - c^3)(c^3 - a^3) \equiv 0 \text{ mod } 7$ for all choices of $a, b, c$. 

2
7. Given a regular polygon of \( n \) sides, with vertices \( A_1, A_2, \ldots, A_n \) and with centre \( O \), construct incentres for each of the triangles \( A_1O_A_2, A_2O_A_3, \ldots, A_nO_A_1 \), calling them \( B_1, B_2, \ldots, B_n \) respectively. Show that the area of the star-shaped region \( A_1B_1A_2B_2 \cdots A_nB_n \) is twice that of a similar regular polygon with side length \( A_1B_1 \). (The incentre of a triangle is the meeting-point of the lines bisecting the internal angles of the triangle. Alternatively, and equivalently, it is the centre of the circle interior to the triangle just touching each of the three sides.)

Let \( D \) be the foot of the perpendicular from \( O \) to \( A_1A_2 \); since \( \triangle OA_1A_2 \) is isosceles this passes through \( B_1 \). Let the length \( A_1B_1 \) equal \( x \) and \( \angle B_1A_1A_2 = \frac{\pi}{4} - \frac{\pi}{2n} = \alpha \). Then if \( r \) is the radius of the incircle, simple geometry gives \( r = x \sin(\alpha) \). The length of \( A_1D \) is given by \( \frac{A_1D}{\sin(\angle OA_1A_2)} = \frac{x \cos(\alpha)}{\cos(2\alpha)} \). Hence, the area of the star-shaped region is \( 2n|\triangle OA_1B_1| = nr \cdot A_1O = nx \sin(\alpha) \frac{x \cos(\alpha)}{\cos(2\alpha)} = \frac{1}{2}nx^2 \tan(2\alpha) \).

The area of the whole polygon is equal to \( n \) times the area of \( \triangle OA_1A_2 \), or \( \frac{1}{2}A_1A_2 \cdot OD \). Since \( OD = A_1D \tan(2\alpha) = \frac{1}{2}A_1A_2 \tan(2\alpha) \), the area of the polygon is \( \frac{1}{8}(A_1A_2)^2 \tan(2\alpha) \).

The area of the polygon with side \( x \) is therefore found by replacing \( A_1A_2 \) with \( x \) to give \( \frac{1}{8}nx^2 \tan(2\alpha) \), which completes the proof.

8. You are given 2016 real numbers \( x_1, x_2, \ldots, x_n \) all lying in the interval \(-1 \leq x_i \leq 1\). The sum of their cubes is zero. Find the maximum sum of the numbers.

Note that the inequality \( 4x^3 - 3x + 1 = (x + 1)(2x - 1)^2 \geq 0 \) is true for all \( x \geq -1 \). So

\[
\sum_{k=1}^{2016} (4x_k^3 - 3x_k + 1) = 2016 - 3 \sum_{k=1}^{2016} x_k \geq 0.
\]

From this follows the result \( \sum_{k=1}^{2016} x_k \leq 2016/3 = 672 \). This can be attained by letting \( x_k = -1 \) for \( 1 \leq k \leq 224 \) and \( x_k = \frac{1}{2} \) for \( 225 \leq k \leq 2016 \), and hence the answer is 672.