The University of Melbourne–School of Mathematics and Statistics
School Mathematics Competition, 2017

INTERMEDIATE DIVISION: SOLUTIONS

(1) A polygon has all interior angles equal to 130 degrees, with the exception of one angle. List all of the possible values of the exceptional angle.

Solution. An *exterior angle* of a polygon is 180 - interior angle. So most of the exterior angles of the polygon here have exterior angle $180 - 130 = 50$ degrees. The sum of the exterior angles of a polygon is always 360 degrees. We have

$$\begin{aligned} 360 &= 7 \times 50 + 10 = 6 \times 50 + 60 = 5 \times 50 + 110 = 4 \times 50 + 160 = 3 \times 50 + 210 \\ &= 8 \times 50 - 40 = 9 \times 50 - 90 = 10 \times 50 - 140 = 11 \times 50 - 190. \end{aligned}$$

An exterior angle can be negative when the polygon is non-convex. (We are grateful to Zefeng Li for pointing out an error in the original solution.) Here the leftover angle is the exceptional exterior angle which must lie between -180 and 180, hence 210 or greater and -190 and smaller do not work, and we are left with the seven possibilities of exceptional exterior angles 160, 110, 60, 10, -40, -90, -140 hence exceptional interior angles 20, 70, 120, 170, 220, 270, 320 degrees.

(2) A train travels at 7 times the speed of a runner. If it takes 8 seconds for the entire train to pass the runner when they are traveling in the same direction, how long will it take for the entire train to pass the runner when they are traveling in opposite directions?

Solution. Let the runner have speed R and let the train have speed $T = 7R$ and length L. If it takes $t_1 = 8$ seconds for the entire train to pass the runner when they are traveling in the same direction, and t_2 seconds for the entire train to pass the runner when they are traveling in opposite directions then

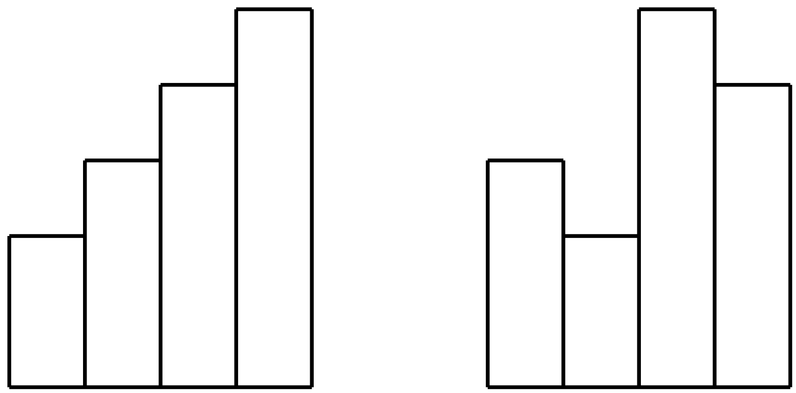
$$t_1 = \frac{L}{T - R}, \quad t_2 = \frac{L}{T + R}$$

so

$$\frac{t_2}{t_1} = \frac{T - R}{T + R} = \frac{T/R - 1}{T/R + 1} = \frac{7 - 1}{7 + 1} = \frac{3}{4}$$

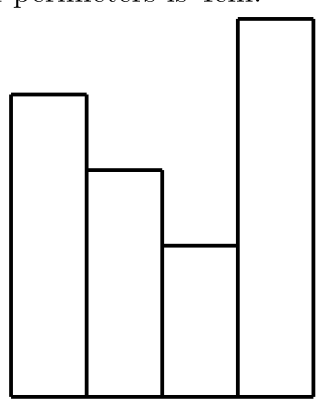
hence $t_2 = \frac{3}{4} t_1 = 6$ seconds.

(3) Consider four rectangles of dimensions 1cm$\times$2cm, 1cm$\times$3cm, 1cm$\times$4cm and 1cm$\times$5cm stacked next to each other along a 4cm horizontal base. Two possible ways to stack them are shown in the diagram. What is the difference between the shortest and longest possible perimeters of such arrangements?



Solution. The minimum perimeter equals the perimeter of the 5×4 rectangle and is realised by any convex arrangement such as in the diagram on the left.

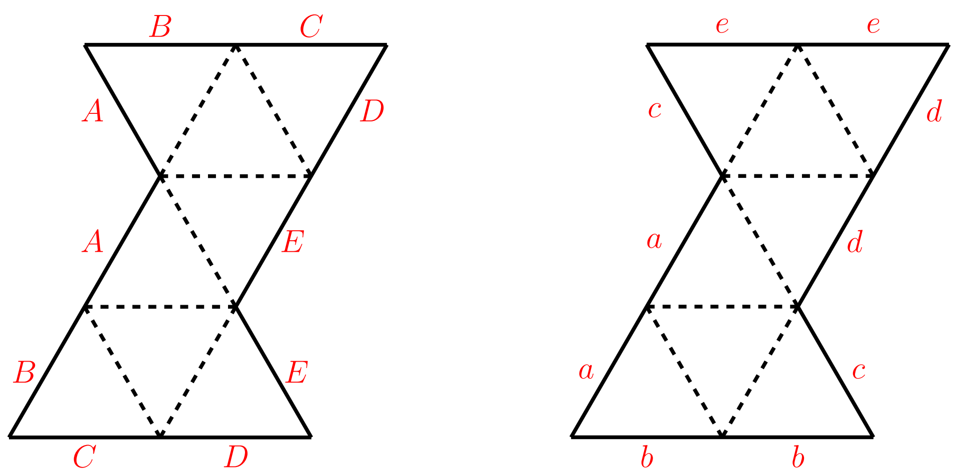
For maximum perimeter, it is easy to argue that the longest rectangle can be placed at one end, leaving six possibilities. The diagram shows the maximum perimeter arrangement and the difference of perimeters is 4cm.



(4) Each day a boy meets his mother at the train station after school and then she drives him home. She always arrives exactly on time to pick him up. One day he catches an earlier train and arrives at the station one hour earlier. He immediately begins walking home along the same route his mother drives, both traveling at constant speed. Eventually, his mother sees him on her way to the station and drives him the rest of the way home. When they arrive home, the boy notices that they arrived 30 minutes earlier than usual. How much time did the boy spend walking?

Solution. The mother turns the car around 15 minutes before she is due to arrive at the station because she saves 30 minutes = 15 minutes in each direction. Since she aimed to arrive one hour later than the boy began to walk, he had been walking for 45 minutes before she reached him.

(5) The net below is folded in two different ways to form two solid shapes. The letters in each diagram show which edges must meet, so for example edge A is glued to edge A. If the volumes enclosed by each of the two solid shapes are given by V for the diagram on the left, and v for the diagram on the right, then calculate V/v.



Solution. The solid shape on the left can also be obtained by beginning with a pyramid of volume P with each of the four faces given by an equilateral triangle, made up of four of the equilateral triangles in the diagram, and removing four pyramids of half the size hence volume $P/8$, one from each of the four corners of the large pyramid. The volume V of this shape is $P - 4P/8 = P/2$.

The solid shape on the right is simply two half-sized pyramids meeting along an edge. Hence the volume is $v = P/8 + P/8 = P/4$. Thus $V/v = 2$.

(6) Find all positive integers n satisfying

$$n = 11111 \times (\text{sum of the digits of } n).$$

Solution. In fact n has a factor of 9, i.e. $n = 99999m$ explained as follows. In words, if n has a remainder after division by 9 (e.g. remainder 1 or 2) then $11111\times$ sum of the digits of n has 5 times that remainder (e.g. remainder 5 or $5 \times 2 = 10 \equiv 1$). But this is a contradiction unless the remainder is 0.

More formally, put $\sigma(n)$ =sum of the digits of n. Then $\sigma(n) \equiv n (\text{mod } 9)$ (since $10 \equiv 1(\text{mod } 9)$, $100 \equiv 1(\text{mod } 9)$, $1000 \equiv 1(\text{mod } 9)$, etc.) Since $11111 \equiv 5(\text{mod } 9)$ then $n = 11111\sigma(n) \equiv 5n(\text{mod } 9)$ hence $n \equiv 0(\text{mod } 9)$.

We will show that $n = 99999m$ for $1 \leq m \leq 10$. If n has k digits then $\sigma(n) \leq 9k$ hence

$$n = 11111\sigma(n) \leq 99999k < 10^{k-1}, \quad k > 6$$

where the last inequality comes from: $99999k < 10^5k < 10^{k-1} \Leftarrow k < 10^{k-6} \Leftarrow k > 6$. In other words, for $k > 6$, $99999k$ is less than every k digit number, hence $n \leq 99999k < n$ which is a contradiction. Thus $n = 99999m$ has at most 6 digits hence $1 \leq m \leq 10$. Now it is easy to see that $\sigma(99999m) = 45$ for $1 \leq m \leq 10$ hence $n = 45 \times 11111 = 499995$.