The University of Melbourne–Department of Mathematics and Statistics
School Mathematics Competition, 2017

JUNIOR DIVISION

Time allowed: Two hours

These questions are designed to test your ability to analyse a problem and to express yourself clearly and accurately. The following suggestions are made for your guidance:

(1) *Considerable weight will be attached by the examiners to the method of presentation of a solution. Candidates should state as clearly as they can the reasoning by which they arrived at their results. In addition, more credit will be given for an elegant than for a clumsy solution.*

(2) *The* ***six*** *questions are not of equal length or difficulty. Generally, the later questions are more difficult than the earlier questions.*

(3) *It may be necessary to spend considerable time on a problem before any real progress is made.*

(4) *You may need to do considerable rough work but you should then write out your final solution neatly, stating your arguments carefully.*

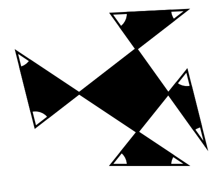
(5) *Credit will be given for partial solutions; however a good answer to one question will normally gain you more credit than sketchy attempts at several questions.*

Textbooks, electronic calculators and computers are ***NOT*** *allowed. Otherwise normal examination conditions apply.*

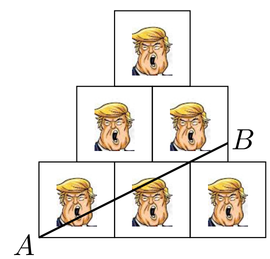
1. Darcy Vescio. Carlton's Darcy Vescio took out the inaugural award for the AFLW's Leading Goalkicker. Darcy kicked three more goals than runner up Sarah Perkins (Crows), but 6 fewer goals than Sarah and Alyssa Mifsud (Demons) combined. In total, Darcy, Sarah and Alyssa kicked 34 goals for the season. How many goals did Darcy kick?

2. Year of the (fire) rooster. In Chinese astrology, each year is associated with one of 12 animals as well as one of 5 elements: earth, fire, gold, water and wood. Currently, we are in the year of the fire rooster. According to Chinese astrology, a character trait of fire roosters is trustworthiness: they always tell the truth, unlike snakes, who always lie. A total of 2017 snakes and fire roosters form a long queue. Each animal, with the exception of the one at the front of the queue, claims that they are "standing" directly behind a snake. The animal at the front of the queue claims that all other animals are snakes. What is the total number of fire roosters in the queue?

3. Pablo Picasso. Pablo Picasso (1881–1973) is widely regarded as one of the most important painters of the last century. He was also a successful sculptor. His largest sculpture, the "Chicago Picasso", is over 15 metres high and weighs close to 150 tons. An artist's impression of the Chicago Picasso, which comprises four triangles and one quadrilateral, is shown on the right. What is the sum of the eight angles shown in white in Picasso's sculpture?



4. (S)he doesn't cut it. For his 70th birthday (held on June 14, 2016), Melania baked Donald a cake consisting of six squares, representing (some would say overestimating) his mental age. A photo of the cake, leaked by the fake media, is shown on the right. With one single cut Melania wishes to cut the cake into two equal-sized pieces, as is shown in the (failed) example on the right. If all squares have side length 10cm and Melania cuts from A on the bottom left corner to B on the right of the cake, determine the exact height of point B so that the cake is cut into two equal-sized pieces.



5. Malcolm in the middle. It is widely known that our Prime Minister, Malcolm Turnbull, loves to travel by public transport. However, as a Sydneysider, his knowledge of Melbourne's bus fares is limited. During a recent visit to Melbourne he took a bus trip from Carlton to Collingwood. He knew that the fare was at least \$3.00, but less than \$5.00. He also knew that the exact fare was required. What is the minimum number of coins (remember we have 5c, 10c, 20c, 50c, \$1 and \$2 coins) that he should have taken with him to be able to pay the exact fare?

6. Lucky numbers. You and your family attend a fair at the local primary school. Each family member buys exactly n raffle tickets, where n is some whole number greater than two. On each ticket a random positive whole number is printed. Your Mum, Dad, brother and sister are all amazed to discover that they can select three of their n tickets such that the sum of the numbers printed on these three is a multiple of 3. You explain to your family members that this is not amazing at all but a hard mathematical fact. What is the smallest value of n for which this is true, and why?

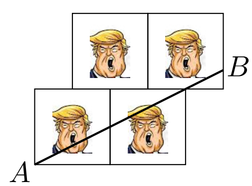
Solutions

1. Darcy Vescio. Since Darcy kicked 3 more goals than Sarah but 6 fewer than Sarah and Alyssa combined, Alyssa must have scored $6+3=9$ goals. Given the total number of goals was 34 this means that Darcy and Sarah kicked 25 goals combined. Hence Darcy kicked 14 and Sarah 11.

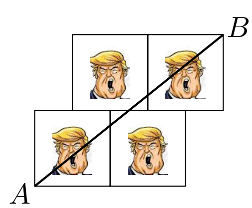
2. Year of (fire) rooster. Label the animals from 1 to 2017 with number 1 standing at the front of the queue and number 2017 at the back of the queue. Number 1 cannot be telling the truth. If all others in the queue were indeed snakes they would all lie and claim (with the exception of number 2) that they were standing behind a rooster. Hence number 1 is a snake. Number 2 (who claimed to be standing behind a snake) is thus telling the truth and must be a rooster. Number 3 (who claimed to be standing behind a snake) is thus telling a lie and must be snake. Number 4 (who claimed to be standing behind a snake) is thus telling the truth and must be a rooster. Proceeding as above we conclude that all even-numbered animals are roosters and all odd-numbered animals are snakes. There are thus 1008 roosters and 1009 snakes in the queue.

3. Pablo Picasso. The sum of the angles of each triangle is $180°$. The sum of the angles of a quadrilateral in the middle of the sculpture is $360°$. But each of the angles of the quadrilateral is the same as the unmarked angle of a the corresponding triangle. Hence the four unmarked angles of the triangles sum up to $360°$. The total of the marked angles is thus $4 \times 180° - 360° = 360°$.

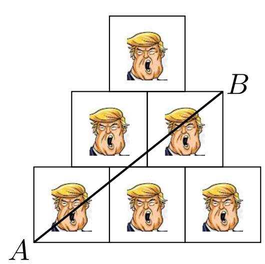
4. (S)he doesn't cut it. Clearly all of the top square will belong to one piece and all of the bottom-right square will belong to the other piece. Hence we may remove both without changing the problem:



But now the problem is perfectly symmetric so that the cut must be:



Putting back the removed two squares this looks like:



The height of B should thus be 20cm.

5. Malcolm in the middle. Eight coins will suffice, for example one of each coin, plus a second 2 dollar coin and a second 20 cent coin as is demonstrated below:

$\$3.00 = \$2 + \$1$	$\$4.00 = 2 \times \$2$
$\$3.05 = \$2 + \$1 + 5c$	$\$4.05 = 2 \times \$2 + 5c$
$\$3.10 = \$2 + \$1 + 10c$	$\$4.10 = 2 \times \$2 + 10c$
$\$3.15 = \$2 + \$1 + 10c + 5c$	$\$4.15 = 2 \times \$2 + 10c + 5c$
$\$3.20 = \$2 + \$1 + 20c$	$\$4.20 = 2 \times \$2 + 20c$
$\$3.25 = \$2 + \$1 + 20c + 5c$	$\$4.25 = 2 \times \$2 + 20c + 5c$
$\$3.30 = \$2 + \$1 + 20c + 10c$	$\$4.30 = 2 \times \$2 + 20c + 10c$
$\$3.35 = \$2 + \$1 + 20c + 10c + 5c$	$\$4.35 = 2 \times \$2 + 20c + 10c + 5c$
$\$3.40 = \$2 + \$1 + 2 \times 20c$	$\$4.40 = 2 \times \$2 + 2 \times 20c$
$\$3.45 = \$2 + \$1 + 2 \times 20c + 5c$	$\$4.45 = 2 \times \$2 + 2 \times 20c + 5c$
$\$3.50 = \$2 + \$1 + 50c$	$\$4.50 = 2 \times \$2 + \$1 + 50c$
$\$3.55 = \$2 + \$1 + 50c + 5c$	$\$4.55 = 2 \times \$2 + 50c + 5c$
$\$3.60 = \$2 + \$1 + 50c + 10c$	$\$4.60 = 2 \times \$2 + 50c + 10c$
$\$3.65 = \$2 + \$1 + 50c + 10c + 5c$	$\$4.65 = 2 \times \$2 + 50c + 10c + 5c$
$\$3.70 = \$2 + \$1 + 50c + 20c$	$\$4.70 = 2 \times \$2 + 50c + 20c$
$\$3.75 = \$2 + \$1 + 50c + 20c + 5c$	$\$4.75 = 2 \times \$2 + 50c + 20c + 5c$
$\$3.80 = \$2 + \$1 + 50c + 20c + 10c$	$\$4.80 = 2 \times \$2 + 50c + 20c + 10c$
$\$3.85 = \$2 + \$1 + 50c + 20c + 10c + 5c$	$\$4.85 = 2 \times \$2 + 50c + 20c + 10c + 5c$
$\$3.90 = \$2 + \$1 + 50c + 2 \times 20c$	$\$4.90 = 2 \times \$2 + 50c + 2 \times 20c$
$\$3.95 = \$2 + 50c + 2 \times 20c + 5c$	$\$4.95 = 2 \times \$2 + 50c + 2 \times 20c + 5c$

Of course your solution does not need to include this complete list of possibilities. The second column is simply the same as the first in which a one dollar coin has been replaced by a two dollar coin, and the second half of each column is the first half to which a 50 cent piece has been added. Looking at \$3.85 and \$4.85, which both need a minimum of 6 coins, but not the same 6 coins (since otherwise \$3.85 would have to be equal to \$4.85) shows that you cannot do with fewer than 7 coins. It also shows that to have any chance to achieving all possible fares with 7 coins we need at least one of each coin, and either a second 2 dollar coin or a second 1 dollar coin (since \$4.85 is as above or $\$4.85 = \$2 + 2 \times \$1 + 50c + 20c + 10c + 5c$). Both these sets of seven coins cannot produce \$3.95 or \$4.95, ruling out that 7 coins is enough.

6. Lucky numbers. Four numbers are not enough, for example, $1, 2, 4, 5$ do not allow you to select three numbers whose sum is a multiple of three (the possible triple-sums are $1 + 2 + 4 = 7$, $1 + 2 + 5 = 8$, $1 + 4 + 5 = 10$ and $2 + 4 + 5 = 11$. However 5 number will be enough as can be seen as follows. Since we are interested to find out if a sum of three numbers is a multiple of three we do not care about the actual value of the numbers making up that sum but only about their remainder after division by three. The numbers $3, 6, 9, 12, 15, \ldots$ are in this sense all equivalent, and so are $1, 4, 7, 10, 13, \ldots$ and $2, 5, 8, 11, 14, \ldots$. Indeed, since $1+2+3$ is divisible by three, so is any sum you obtain by replacing 1 by one of $4, 7, 10, 13, \ldots$, and/or 2 by one of $5, 8, 11, 14, \ldots$ and/or 3 by one of $6, 9, 12, 15, \ldots$. Write a number in the list $3, 6, 9, 12, 15, \ldots$ as 0 (since that is its remainder after division by 3. Similarly, write a number in the list $1, 4, 7, 10, 13, \ldots$ as 1 and write a number in the list $2, 5, 8, 11, 14, \ldots$ as 2.

Now, if you have three 0's (i.e., three numbers divisible by 3) then their sum is also 0 (i.e., also divisible by 3). If you have only two zeros, you must have one of: (i) $\{0, 0, 1, 1, 1\}$: $1 + 1 + 1 = 3 = 0$. (ii) $\{0, 0, 1, 1, 2\}$: $0 + 1 + 2 = 3 = 0$. (iii) $\{0, 0, 1, 2, 2\}$: $0 + 1 + 2 = 3 = 0$. (iv) $\{0, 0, 2, 2, 2\}$: $2 + 2 + 2 = 6 = 0$. If you have one zero, you must either have at least three 1's (form $1 + 1 + 1 = 3 = 0$) or at least three 2's (form $(2 + 2 + 2 = 6 = 0)$, or at least a 1 and a 2 (form $0 + 1 + 2 = 3 = 0$). If you have no 0's you must either have three 1's (form $1 + 1 + 1 = 3 = 0$) or three 2's (form $2 + 2 + 2 = 6 = 0$).