The University of Melbourne–Department of Mathematics and Statistics
School Mathematics Competition, 2017

SENIOR DIVISION

Time allowed: Three hours

These questions are designed to test your ability to analyse a problem and to express yourself clearly and accurately. The following suggestions are made for your guidance:

(1) *Considerable weight will be attached by the examiners to the method of presentation of a solution. Candidates should state as clearly as they can the reasoning by which they arrived at their results. In addition, more credit will be given for an elegant than for a clumsy solution.*

(2) *The **eight** questions are not of equal length or difficulty. Generally, the later questions are more difficult than the earlier questions.*

(3) *It may be necessary to spend considerable time on a problem before any real progress is made.*

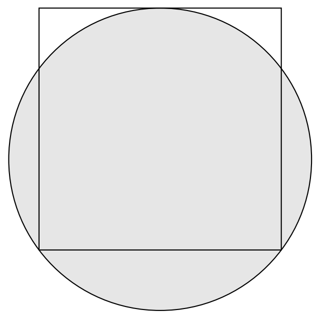
(4) *You may need to do considerable rough work but you should then write out your final solution neatly, stating your arguments carefully.*

(5) *Credit will be given for partial solutions; however a good answer to one question will normally gain you more credit than sketchy attempts at several questions.*

*Textbooks, electronic calculators and computers are **NOT** allowed. Otherwise normal examination conditions apply.*

Q1 Given $x + y = 1$, and $x^2 + y^2 = 2$, evaluate $x^3 + y^3$.

Q2. A circle touches one side of a square and passes through the vertices of the opposite side, as shown in the figure below. Which is longer – the perimeter of the square or the circumference of the circle? (You can use the approximation $\pi \approx 22/7$ in this calculation).



Q3. A perfect number is a number that is equal to the sum of its proper divisors. For example, $28 = 1+2+4+7+14$ is perfect. Note that $28 = 2^2 \cdot 7 = 2^2(2^3 - 1)$. It turns out that if $2^{n+1} - 1$ is prime, then $2^n(2^{n+1} - 1)$ is perfect. Prove this.

Q4. Prove that every polyhedron (a polyhedron is a solid in three dimensions with flat polygonal faces, straight edges and sharp corners or vertices, such as a cube, a pyramid, a dodecahedron) has two faces with the same number of vertices.

Q5. The internal angles of a triangle are α, β, and γ. Prove that

$$1 \le \cos\alpha + \cos\beta + \cos\gamma \le 3/2.$$

Q6. Bob and Mary play an (unfair) game in which Bob starts by rolling a fair, six-sided dice, and then Mary tosses a fair, two-sided coin. They repeat this alternating pattern until one of them wins. Bob wins if he rolls a 6, while Mary wins if she tosses a head. What is the probability that Bob wins the game?

Q7. Prove that

$$\sqrt{3}(\sqrt{3} - 1)^n - (-1)^n\sqrt{3}(\sqrt{3} + 1)^n$$

is divisible by 6 for all positive integers n.

Q8. A finite number of points in the plane are chosen, and each is coloured either red or blue. Suppose that for any line l in the plane, the difference between the number of red points on l and the number of blue points on l is at most 1. Prove that the points all lie on a single line.