The University of Melbourne–School of Mathematics and Statistics School Mathematics Competition, 2017

SENIOR DIVISION

Q1. Given $x + y = 1$, and $x^2 + y^2 = 2$, evaluate $x^3 + y^3$.

Solution.

$$x^3 + y^3 = (x + y)^3 - 3x^2y - 3xy^2 = (x + y)^3 - 3xy(x + y) = 1 - 3xy.$$

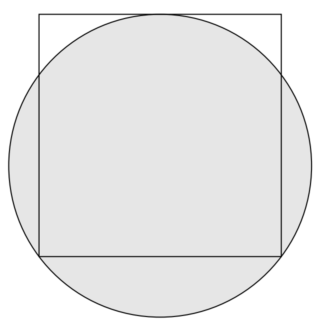
Now,

$$xy = \frac{1}{2}\left((x + y)^2 - (x^2 + y^2)\right) = -\frac{1}{2},$$

so

$$x^3 + y^3 = \frac{5}{2}.$$

Q2. A circle touches one side of a square and passes through the vertices of the opposite side, as shown in the figure below. Which is longer – the perimeter of the square or the circumference of the circle? (You can use the approximation $\pi \approx 22/7$ in this calculation).



Solution. Label the two bottom vertices of the square C and D, the mid-point of the line joining them as B and the centre of the circle as O. Let the side of the square (the length of the segment CD) be $2a$, and the radius of the circle be r. Then $OD = OC = r$, $OB = 2a - r$, and $BC = a$. So by Pythogoras' theorem we have for the triangle OBC that $r^2 = a^2 + (2a - r)^2$.

From this equation it follows that

$$\frac{a}{r} = \frac{4}{5}.$$

The ratio of the circumference of the square to that of the circle is

$$\frac{8a}{2\pi r} = \frac{32}{10\pi} = \frac{32}{31.4159\cdots} > 1,$$

so the square has a greater circumference.

Q3. A perfect number is a number that is equal to the sum of its proper divisors. For example, $28 = 1 + 2 + 4 + 7 + 14$ is perfect. Note that $28 = 2^2 \cdot 7 = 2^2(2^3 - 1)$. It turns out that if $2^{n+1} - 1$ is prime, then $2^n(2^{n+1} - 1)$ is perfect. Prove this.

Solution. Let $s(m)$ be the sum of the proper divisors of m. So m is perfect if $s(m) = m$. Let $p = 2^{n+1} - 1$. If p is prime, then the only proper divisors of $m(= 2^n p)$ are:

$$1, 2, 2^2, \ldots, 2^n, p, 2p, \ldots 2^{n-1}p.$$

So

$$s(m) = (1 + p)(1 + 2 + \ldots + 2^n) - 2^n p = 2^{n+1}(2^{n+1} - 1) - 2^n p = 2^n p = m.$$

Q4. Prove that every polyhedron (a polyhedron is a solid in three dimensions with flat polygonal faces, straight edges and sharp corners or vertices, such as a cube, a pyramid, a dodecahedron) has two faces with the same number of vertices.

Solution Let P be a polyhedron, and let L be a face of P with a maximal number n of edges. Then L shares an edge with n other faces. Each of these faces has at least three and at most n edges. Therefore by the pigeon-hole principle, there must be two of them that have the same number of edges.

Q5. The internal angles of a triangle are α, β, and γ. Prove that

$$1 \leq \cos\alpha + \cos\beta + \cos\gamma \leq 3/2.$$

Solution Label the sides of the triangle a, b, c. Assume, without loss of generality, $a \leq b \leq c$. Now $\cos\alpha = \frac{b^2+c^2-a^2}{2bc}$, and similarly $\cos\beta = \frac{a^2+c^2-b^2}{2ac}$, $\cos\gamma = \frac{b^2+a^2-c^2}{2ba}$. So

$$T = \cos\alpha + \cos\beta + \cos\gamma = \frac{1}{2abc}[ab^2 + ac^2 + bc^2 + ba^2 + ca^2 + cb^2 - a^3 - b^3 - c^3].$$

So

$$T = 1 + \frac{1}{2abc}(b + c - a)(a + c - b)(a + b - c) \geq 1,$$

since $(b+c-a) > 1$ etc. by the triangle inequality.
Another factorisation is

$$T = \frac{a(b-c)^2 - a^3 + b(c-a)^2 - b^3 + c(a-b)^2 - c^3 + 6abc}{2abc}$$

$$= \frac{3}{2} + \frac{1}{2abc}\left([a^2(b+c-a) - abc] + [b^2(a+c-b) - abc] + [c^2(b+a-c) - abc]\right)$$

$$= \frac{3}{2} - \frac{1}{2abc}\left(a(a-b)(a-c) + b(b-a)(b-c) + c(c-a)(c-b)\right).$$

Recalling $a \le b \le c$, it follows that $(a-b)(a-c) \ge 0$. Further, $b(b-a)(b-c)+c(c-a)(c-b) = (c-b)^2(c+b-a) \ge 0$. Hence $T \le \frac{3}{2}$.

Q6. Bob and Mary play an (unfair) game in which Bob starts by rolling a fair, six-sided dice, and then Mary tosses a fair, two-sided coin. They repeat this alternating pattern until one of them wins. Bob wins if he rolls a 6, while Mary wins if she tosses a head. What is the probability that Bob wins the game?

Solution On a single roll, Bob has a probability $\frac{1}{6}$ of winning, while Mary has a probability $\frac{1}{2}$ of winning. The probability that Bob wins on the kth roll is the probability that they both lose on the first $(k-1)$ plays and that he tosses a 6 on the kth roll. That is,

$$p_k = \frac{1}{6}\left(\frac{5}{6} \cdot \frac{1}{2}\right)^{k-1}.$$

The overall probability that Bob wins is the sum of this probability over all values of k. This is just a geometric progression, so

$$\sum_{k \ge 1} p_k = \frac{1}{6} \sum_{k \ge 0} \left(\frac{5}{12}\right)^k = \frac{2}{7}.$$

Q7. Prove that

$$\sqrt{3}(\sqrt{3}-1)^n - (-1)^n \sqrt{3}(\sqrt{3}+1)^n$$

is divisible by 6 for all positive integers n.

Solution Recall the binomial theorem:

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \cdots + b^n.$$

When $n = 2m$ is even, we have

$$\sqrt{3}\left((\sqrt{3}-1)^{2m} - (\sqrt{3}+1)^{2m}\right).$$

Expanding this by the binomial theorem, we see that the even-order terms cancel, and the odd-order terms are added. As a result,

$$\begin{aligned} &\sqrt{3}\left((\sqrt{3}-1)^{2m}-(\sqrt{3}+1)^{2m}\right) \\ &= -2\sqrt{3}\left(\tbinom{2m}{1}(\sqrt{3})^{2m-1}+\tbinom{2m}{3}(\sqrt{3})^{2m-3}+\tbinom{2m}{5}(\sqrt{3})^{2m-5}+\cdots+\tbinom{2m}{2m-1}\sqrt{3}\right) \\ &= -6\left(\tbinom{2m}{1}(3)^{m-1}+\tbinom{2m}{3}(3)^{m-2}+\tbinom{2m}{5}(3)^{m-3}+\cdots+\tbinom{2m}{2m-1}3\right). \end{aligned}$$

Every term in the large brackets is an integer, so the result is divisible by 6. When n is odd, set $n = 2m+1$ and the odd terms cancel, and a virtually identical calculation yields the same divisibility result.

More elegantly: Multiply f_n by $(\sqrt{3}-1)$ as well as by $(-\sqrt{3}-1)$ and add the results. This immediately shows that $f_n = -2(f_{n-1} - f_{n-2})$. Since $f_0 = 0$ and $f_1 = 6$ are both divisible by 6 this shows that all f_n are divisible by 6.

Q8. A finite number of points in the plane are chosen, and each is coloured either red or blue. Suppose that for any line l in the plane, the difference between the number of red points on l and the number of blue points on l is at most 1. Prove that the points all lie on a single line. (Problem and solution due to Andrew Elvey Price).

Solution For a fixed number of coloured points n, we proceed by calculating bounds on the number of pairs of points of different colours in two different ways. Call a pair of points *fun* if one is blue and the other is black. First, let r be the total number of red points. Then there are $n-r$ blue points, so the total number of fun pairs of points is $r(n-r)$. For fixed n, this is maximised when $r = n/2$, so there are at most $n^2/4$ fun pairs of points. The total number of pairs of coloured points is given by $\binom{n}{2} = \frac{n(n-1)}{2}$. Hence, amongst all such pairs, the proportion of pairs which are fun is at most

$$\frac{n^2}{4}\Big/\frac{n(n-1)}{2} = \frac{n}{2(n-1)}.$$

Now, let l be any line in the plane which passes through at least two coloured points. Then either l passes through the same number a of blue points and red points, or the number of blue points on l differs from the number of red points on l by 1. If l passes through a points of each colour, then it contains a^2 fun pairs of points out of a total of $\binom{2a}{2}$ pairs. Hence, amongst all pairs of coloured points on l, the proportion of pairs which are fun is exactly

$$a^2\Big/\frac{2a(2a-1)}{2} = \frac{a}{2a-1}.$$

Otherwise, let a be the number of points on l of one colour and let $a-1$ be the number of points on l of the other colour. Then l contains $a(a-1)$ fun pairs of points out of a total of $\binom{2a-1}{2}$ pairs. Hence, amongst all pairs of coloured points on l, the proportion of pairs which are fun is exactly

$$a(a-1)\Big/\frac{(2a-1)(2a-2)}{2} = \frac{a}{2a-1}.$$

Since every pair of coloured points lies on exactly one line, it is impossible for the proportion of fun pairs of points on every line to be strictly below the overall proportion of fun pairs of points. hence for some line l which passes through a points of one colour and a or $a-1$ points of the other colour, we must have

$$\frac{a}{2a-1} \leq \frac{n}{2(n-1)}.$$

Rearranging this gives the inequality $2a \geq n$. But the number of coloured points on l is at least $2a - 1 \geq n - 1$. Hence there is at most one coloured point which is not on l. If there is one such point X, then we can draw a line t through X and any point on l which shares the same colour as X. But then t passes through 2 points of one colour and no points of the other colour - a contradiction. Hence all coloured points lie on l.