The University of Melbourne–School of Mathematics and Statistics
School Mathematics Competition, 2018

INTERMEDIATE DIVISION

Time allowed: Three hours

These questions are designed to test your ability to analyse a problem and to express yourself clearly and accurately. The following suggestions are made for your guidance:

1. *Considerable weight will be attached by the examiners to the method of presentation of a solution. Candidates should state as clearly as they can the reasoning by which they arrived at their results. In addition, more credit will be given for an elegant than for a clumsy solution.*

2. *The* ***seven*** *questions are not of equal length or difficulty. Generally, the later questions are more difficult than the earlier questions.*

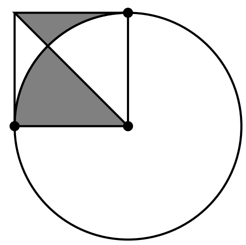
3. *It may be necessary to spend considerable time on a problem before any real progress is made.*

4. *You may need to do considerable rough work but you should then write out your final solution neatly, stating your arguments carefully.*

5. *Credit will be given for partial solutions; however a good answer to one question will normally gain you more credit than sketchy attempts at several questions.*

Textbooks, electronic calculators and computers are ***NOT*** *allowed. Otherwise normal examination conditions apply.*

1. A circle of radius r has its centre at one vertex of a square, as shown in the figure below, and two other vertices on the circumference, as shown. Find the area of the grey region.



2. Mary throws a fair, 6-sided dice. If it comes up greater than 3, she wins. If not, she throws again and if it comes up greater than 4, she wins. Calculate the probability that she wins.

3. In a square of sidelength 3, what is the maximum number of points that can be marked on that square (including the boundary) so that any two are more than $\sqrt{2}$ apart?

4. For which values of n between 2018 and 2099 is $n^6/6!+n^5/5!+n^4/4!+n^3/3!+n^2/2!+n+1$ an integer?

5. The increasing sequence $1, 3, 4, 9, 10, 12, 13, \ldots$ consists of all those positive integers which are either powers of 3 or sums of *distinct* powers of 3. Find the 100^{th} term in the sequence, where 1 is the first term, 3 is the second etc.

6. Suppose that N is a positive, sixteen digit integer. Show that we can find some consecutive digits of N such that the product of these digits is a perfect square.

7. From a group of 23 people, one is chosen as a referee, and the remaining 22 are split into two soccer teams of 11 players in such a way that the total weight of both teams is the same. You may assume that everyone's weight is a whole number of kilos. Show that if this can always be done, regardless of who is chosen as referee, then all 23 players must have the same weight.