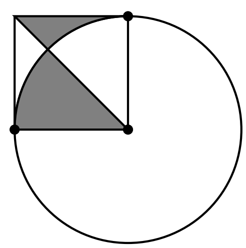
The University of Melbourne–School of Mathematics and Statistics
School Mathematics Competition, 2018

INTERMEDIATE DIVISION SOLUTIONS

1. A circle of radius r has its centre at one vertex of a square, as shown in the figure below, and two other vertices on the circumference, as shown. Find the area of the grey region.



Solution: The shaded regions together are half the area of the square, so the area of the grey (shaded) region is $r^2/2$.

2. Mary throws a fair, 6-sided dice. If it comes up greater than 3, she wins. If not, she throws again and if it comes up greater than 4, she wins. Calculate the probability that she wins.

Solution: The probability is $\frac{3}{6} + \frac{3}{6}\frac{2}{6} = \frac{2}{3}$.

3. In a square of sidelength 3, what is the maximum number of points that can be marked on that square (including the boundary) so that any two are more than $\sqrt{2}$ apart?

Solution: Clearly, in a unit square, any two points are no more than $\sqrt{2}$ apart. Divide a square of side 3 into nine unit squares. If 10 points are placed in the square, by the pigeon-hole principle, two must be in the same unit square, and so less than or equal to $\sqrt{2}$ apart. So 9 is the maximum possible. This is achievable by placing a point at each vertex of the square, a point at each mid-point of each edge, and a point in the middle of the square. No pair of points is closer than $1.5 > \sqrt{2}$.

4. For which values of n between 2018 and 2099 is $n^6/6!+n^5/5!+n^4/4!+n^3/3!+n^2/2!+n+1$ an integer?

Solution: Since $1+n$ is always an integer, we can simplify the problem to ask when is $n^6/6!+n^5/5!+n^4/4!+n^3/3!+n^2/2!$ an integer? Rewrite this as $\frac{n^6+6n^5+30n^4+120n^3+360n^2}{6!}$. So the numerator must be divisible by 6!. All terms in the numerator but the first are always divisible by 6, so n^6 must be divisible by 6, so n must be divisible by 6. This gives 2022, 2028, 2034, 2040, 2046, 2052, 2058, 2064, 2070, 2076, 2082, 2088 and 2094 as contendors. Now, all but the first two terms in the numerator are divisible by 5, so $n^6+6n^5=n^5(n+6)$ must be divisible by 5. Therefore either n is divisible by 5 or $n+6$ is divisible by 5. The first condition gives 2040 and 2070 from the list of contendors, while the second condition adds 2034, 2064 and 2094. Since $\frac{30n^4+120n^3+360n^2}{6!}$ is an integer, this enough to ensure

$$2040,\ 2070,\ 2034, 2064,\ 2094$$

are solutions.

5. The increasing sequence 1, 3, 4, 9, 10, 12, 13, ... consists of all those positive integers which are either powers of 3 or sums of *distinct* powers of 3. Find the 100^{th} term in the sequence, where 1 is the first term, 3 is the second etc.

Solution: Write the numbers from 1 to 100 in binary notation. They are then your first 100 numbers in ternary notation (base 3). So the solution is $(1100100)_3 = 729+243+9 = 981$.

6. Suppose that N is a positive, sixteen digit integer. Show that we can find some consecutive digits of N such that the product of these digits is a perfect square.
Solution: If there is a digit zero, then that is the desired number. If there is no digit 0, let $N_i = 2^{a_i}\cdot 3^{b_i}\cdot 5^{c_i}\cdot 7^{d_i}$ be the product of the digits of N from the first to the i^{th} digit. Each of the four indices are 0 or 1 *modulo* 2. If all four indices are 0 (mod 2),(there are 16 cases in all), then the desired number is the square, A_i. Otherwise we can find some $j>i$, and $N_j > N_i$, such that $a_i = a_j$, $b_i = b_j$, $c_i = c_j$, $d_i = d_j$, (mod 2). The desired square number is then N_j/N_i, which is the product of successive digits from the $i+1$-th to the j-th.

7. From a group of 23 people, one is chosen as a referee, and the remaining 22 are split into two soccer teams of 11 players in such a way that the total weight of both teams is the same. You may assume that everyone's weight is a whole number of kilos. Show that if this can always be done, regardless of who is chosen as referee, then all 23 players must have the same weight.
Solution: Assume the contrary. Then choose the set with minimal total weight. Let $W = w_1+\cdots+w_{23}$ be the total weight, and w_i be the weight of the referee. Then $W-w_i$

is even, and $w_i \equiv W \pmod 2$. So the individual weights have the same parity. If they are all even, we can divide all the weights by 2 and still satisfy the conditions, which violates the assumption of minimality. If they are all odd, we can add 1 to each weight and repeat the previous exercise.

Alternative solution: Suppose not everyone has the same weight. Then let x be the weight of the lightest person. Subtract x from everyone's weights, then the conditions of the problem are still satisfied, but the lightest person now has weight 0. Now, since not everyone had the same weight initially, there remains at least one person with non-zero weight. Divide all weights by 2 until at least one person has an odd weight, say y. Let W be the total weight. Choosing a person with weight 0 to be the referee, we find that W must be even. However, we are now unable to split the teams evenly if we choose y to be referee, as the remaining weight $W - y$ is odd.