Solutions

1. Girl power. We have $d = vt$, where d is distance swum (in metres), v is speed (in metres per second) and t is time (in seconds). For a fixed d (in this case the 100 metres swum by Cate), if we increase the speed by 25%, i.e., by a factor $1.25 = 5/4$ the time will decrease by a factor $4/5$, since $(5/4) \times (4/5) = 1$:

$$d = vt = (\tfrac{5}{4}v) \times (\tfrac{4}{5}t).$$

Since $t - \frac{4}{5}t = 10$ seconds, we find that Cate's time was $t = 50$ seconds.

2. Bad boy. The optimal strategy is for Steve S. to compare different pairs of balls. A worst-case scenario is that after he has compared the first 49 pairs he has not yet found a mismatch. We know that one of the two remaining balls must be tampered with, and taking one of these (it does not matter which) and comparing it against any of the other 98 balls will be enough to determine which one of the two is clean and which one is not. So for us 50 comparisons is enough. However, there is a catch. Steve S. does not yet know for sure that David W. has actually tampered with a ball. Hence for him a worst-case scenario is that he has to compare both of the remaining balls against one of the other 98 balls, requiring 51 comparisons in total.

3. Dominoes. In the bottom-right corner, the 0 must be paired with a 2. In the top-left corner the 0 must thus be paired with the 4. This in turn implies that in the top-right corner, the 4 must be paired with the 1. Thus

2	0	3	1	4
4	4	1	0	2
5	5	1	4	2
2	2	1	2	0

Since the 0 and 2 below the 1 and 4 in the top-right corner cannot be paired, the 2 and the 2 in the right-most column must be paired, so that we have a 2-0 pair in the bottom-right corner:

2	0	3	1	4
4	4	1	0	2
5	5	1	4	2
2	2	1	2	0

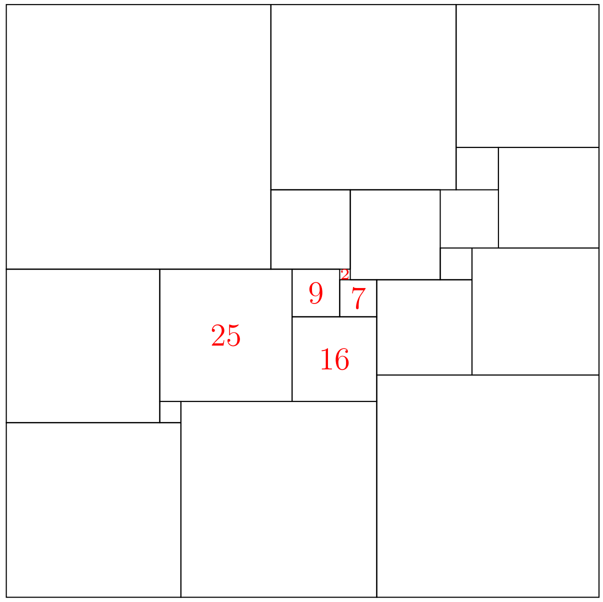
Since we already have a 2-2 domino the bottom-left corner must be 2-5, and to avoid a second 2-5 we must have a 2-1 in the last row. Moreover, since we already have a 1-4 domino we must have a 0-4 next to the 2-2 on the right:

2	0	3	1	4
4	4	1	0	2
5	5	1	4	2
2	2	1	2	0

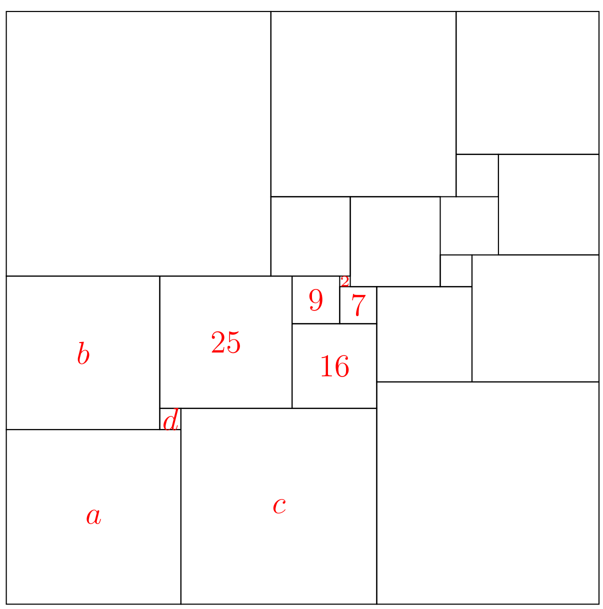
Continuing on, avoiding previously used dominoes, we end up with

2	0	3	1	4
4	4	1	0	2
5	5	1	4	2
2	2	1	2	0

4. Grace. A number of side-lengths can be immediately filled in:



Perhaps a little harder to see is that we can now complete the entire bottom-left section. Indeed if we write



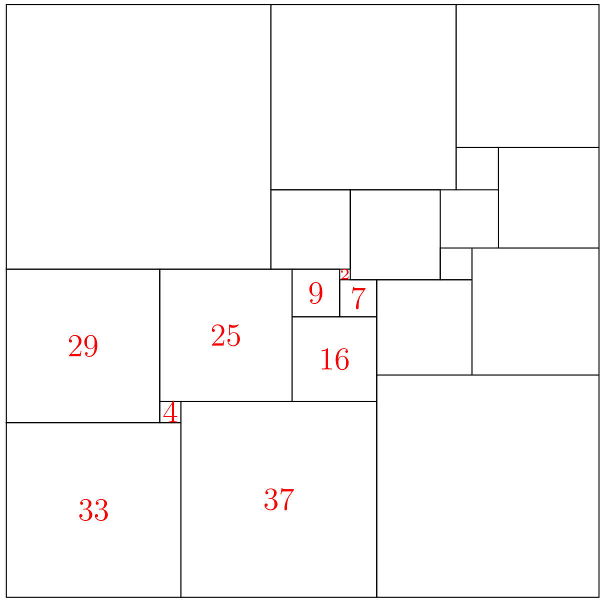
Then, since $25 + 16 = 41$,

$$a + c = b + 41 \quad \text{and} \quad a + b = 25 + c.$$

Taking the sum this is $2a + b + c = 66 + b + c$ so that $a = 33$. This leaves us with $c = b + 8$. But we also have (using that $a = 33$)

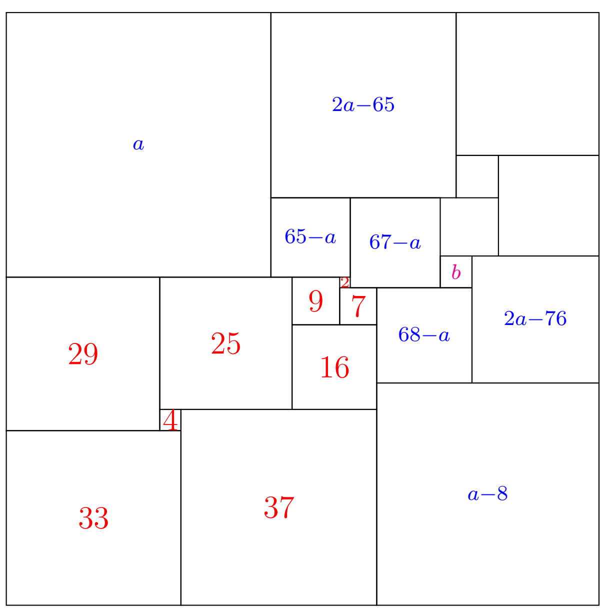
$$d + c = 41 \quad \text{and} \quad d + 33 = c.$$

This gives $c = 37$, $d = 4$ and thus $b = 29$. Hence



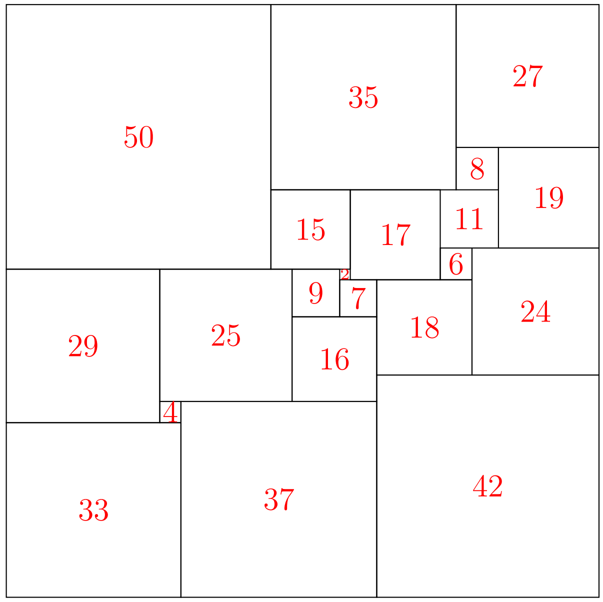
If the side-length of the top-left corner is a, then the total side-length is $62+a$. We can then successively compute the following side-lengths:

$$\begin{aligned} 29+25+9+2-a &= 65-a \\ (65-a)+2 &= 67-a \\ 62-a-(65-a)-62 &= 2a-65 \\ \cdots &= a-8 \\ \cdots &= 2a-76 \\ \cdots &= 68-a \end{aligned}$$



But now b can be computed in two distinct ways: (1) since $a+(65-a)+(67-a)+b+2a-76 = 62+a$ we find $b=6$; (2) since $b=(2a-76)-(68-a)$ we find $b=3a-144$. This implies that $a=50$ so that the total side-length is $62+50=112$.

For the record, the sizes of all of the small squares are given by



5. Year of the dog. We begin by noting that

$$\begin{aligned}1+2+3+\cdots+2018 &= \frac{1}{2}(0+1+2+3+\cdots+2018+2018+\cdots+2+1+0)\\ &= \frac{1}{2}\big((0+2018)+(1+2017)+(2+2016)+\cdots+(2018+0)\big)\\ &= \frac{1}{2}\big(\underbrace{2018+2018+\cdots+2018}_{\text{2019 times}}\big)\\ &= \frac{1}{2}\times 2018\times 2019.\end{aligned}$$

There are 2018 triples of three consecutive numbers, each dog being part of exactly three such triples. Hence the average of the sum of consecutive triples is

$$\frac{3}{2}\times 2018\times 2019\times\frac{1}{2018}=\frac{3}{2}\times 2019=3028+\frac{1}{2}=3028.5.$$

But if all numbers are less than or equal to 3028 then the average cannot be 3028.5. Hence at least one triple must have sum greater or equal to 3029.

Bonus mark: Proceeding by contradiction, we will try to arrange the numbers $1,\ldots,2018$ such that we avoid a triple whose sum exceeds 3029. To achieve this, 1009 of the triples must sum to 3029, otherwise the average will be less than 3028.5. This in turn implies that the other 1009 triples must all sum to 3028. If $A=(a,b,c)$ and $B=(b,c,d)$ are two consecutive triples, then (since $d\neq a$) A and B will have different sum. Hence consecutive triples must alternate between 3028 and 3029. But this is impossible since we cannot have more than 4 consecutive triples alternate between 3029 and 3028, which is a long way short of 2000-plus alternating such triples. To see this, let A,B,C,D,E be a sequences of 5 consecutive triples whose respective sums are 3029, 3028, 3029, 3028, 3029. If $A=(a,b,c)$ (so that $a+b+c=3029$) then $B=(b,c,a-1)$, $C=(c,a-1,b+1)$, $D=(a-1,b+1,c-1)$ and $E=(b+1,c-1,a)$. But a was already used in A so that we cannot have 5 consecutive triples alternate between 3029 and 3028. We conclude that there must be at least one triple exceeding 3029.

6. Blackbeard. The probability of selecting the chest containing rocks and the magic eye-patch then also showing it to contain rocks is $1/3\times 3/4=1/4$. The probability of selecting a chest with treasure and the eye-patch then showing it to contain rocks is $2/3\times 1/4=1/6$. Thus, since $(1/4)/(1/4+1/6)=3/5$, if Blackbeard sees rocks, he has a 3 in 5 (or 60%) chance

that it indeed contains rocks and a 2 in 5 (or 40%) chance that it actually contains treasure. The probability it contains treasure is thus 2/5 or 40%.