The University of Melbourne–Department of Mathematics and Statistics
School Mathematics Competition, 2018

SENIOR DIVISION

Time allowed: Three hours

These questions are designed to test your ability to analyse a problem and to express yourself clearly and accurately. The following suggestions are made for your guidance:

(1) *Considerable weight will be attached by the examiners to the method of presentation of a solution. Candidates should state as clearly as they can the reasoning by which they arrived at their results. In addition, more credit will be given for an elegant than for a clumsy solution.*

(2) *The **seven** questions are not of equal length or difficulty. Generally, the later questions are more difficult than the earlier questions.*

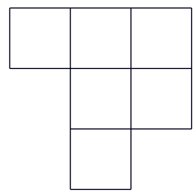
(3) *It may be necessary to spend considerable time on a problem before any real progress is made.*

(4) *You may need to do rough work but you should then write out your final solution neatly, stating your arguments carefully.*

(5) *Credit will be given for partial solutions; however a good answer to one question will normally gain you more credit than sketchy attempts at several questions.*

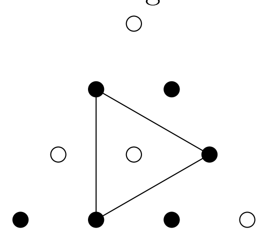
*Textbooks, electronic calculators and computers are **NOT** allowed. Otherwise normal examination conditions apply.*

1. Write a single non-zero digit in each square in the figure so that when reading across and down, the three digit numbers are perfect cubes and the two digit numbers are perfect squares. Find all solutions.

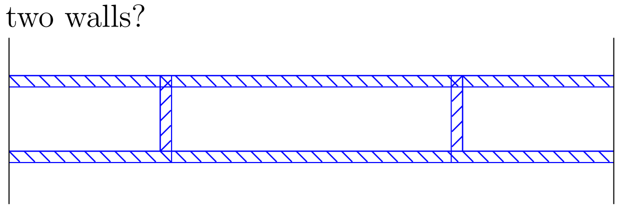


2. Bindy wrote down a binary number. Terry accidentally thought the number was written in base 3. For example, if Bindy wrote 1101, this is equal to $2^3 + 2^2 + 2^0$ which is 13 in decimal notation, whereas Terry would read 1101 as equal to $3^3+3^2+3^0$ which is 37 in decimal notation. When they compared numbers they found that Terry's number was exactly 3 times Bindy's number. What possible numbers can Bindy have written down?

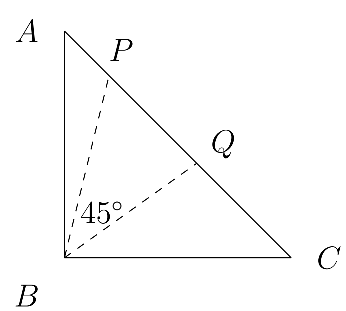
3. Given ten points arranged in a regular triangular array as in the diagram, colour each point either black or white. Prove that there is always an equilateral triangle all of whose vertices have the same colour. An example of a colouring and an equilateral triangle is shown.



4. A rope ladder connects two walls. Each of the 8 sections of the rope that make the ladder is cut with probability $\frac{1}{2}$ independently of the other sections. What is the probability that the ladder still connects the two walls?



5. Consider points P and Q on the hypotenuse $\overline{AC}$ of an isosceles right-angle triangle chosen so that $\angle PBQ = 45°$ as in the figure.



Prove that the three lengths $\overline{AP}$, $\overline{PQ}$ and $\overline{QC}$ form three sides of a new right-angle triangle.

6. List all prime numbers contained in the sequence of 2018 numbers

$$\{101, 10101, 1010101, ..., \overbrace{101010101...1010101}^{2018\ \text{0s}}\}.$$

7. Let r be the maximum possible quotient of the area of a square divided by the area of any equilateral triangle containing that square. Let ρ be the maximum possible quotient of the area of an equilateral triangle divided by the area of any square containing that equilateral triangle. Which of r and ρ is bigger?