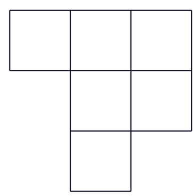
The University of Melbourne–Department of Mathematics and Statistics
School Mathematics Competition, 2018

SENIOR DIVISION: SOLUTIONS

1. Write a single non-zero digit in each square in the figure so that when reading across and down, the three digit numbers are perfect cubes and the two digit numbers are perfect squares. Find all solutions.



Solution: The three digit cubes are 125, 216, 343, 512 and 729. Hence we can have

$$\begin{array}{ccc} 2 & 1 & 6 \\ & 2 & \\ & 5 & \end{array} \qquad \begin{array}{ccc} 5 & 1 & 2 \\ & 2 & \\ & 5 & \end{array} \qquad \begin{array}{ccc} 1 & 2 & 5 \\ & 1 & \\ & 6 & \end{array} \qquad \begin{array}{ccc} 7 & 2 & 9 \\ & 1 & \\ & 6 & \end{array}$$

but the only one that allows us to fill in a square across and down is:

$$\begin{array}{ccc} 5 & 1 & 2 \\ & 2 & 5 \\ & 5 & \end{array}$$

2. Bindy wrote down a binary number. Terry accidentally thought the number was written in base 3. For example, if Bindy wrote 1101, this is equal to $2^3 + 2^2 + 2^0$ which is 13 in decimal notation, whereas Terry would read 1101 as equal to $3^3+3^2+3^0$ which is 37 in decimal notation. When they compared numbers they found that Terry's number was exactly 3 times Bindy's number. What possible numbers can Bindy have written down?

Solution: Assume the number is positive. (One can take 0 and the negative of any answer.) Write $1101_2 = 2^3 + 2^2 + 2^0$ for base 2 (binary) and $1101_3 = 3^3 + 3^2 + 3^0$ for base 3.

The number cannot have more than 5 digits because $3 \times 111111_2 = 3 \times 63 < 243 = 100000_3$ showing that $3\times$ *any* 6 digit number in base 2 is less than *any* 6 digit number in base 3. More generally, for $n > 5$, $3 \times (2^n - 1) < 3^n$.

Since Terry's number is a multiple of 3 it must end in 0 so the binary number is even, hence Terry's number is a multiple of 6—so it ends in zero and has an even number of 1s—with at most five digits. There are seven (non-zero) options.

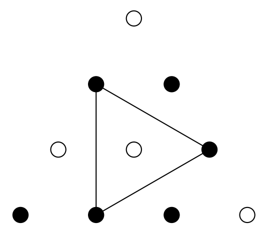
$$6 = 110_2 \mapsto 110_3 = 12, \quad 10 = 1010_2 \mapsto 1010_3 = 30, \quad 12 = 1100_2 \mapsto 1100_3 = 36,$$

$$18 = 10010_2 \mapsto 10010_3 = 84, \quad 20 = 10100_2 \mapsto 10100_3 = 90, \quad 24 = 11000_2 \mapsto 11000_3 = 108,$$

$$30 = 11110_2 \mapsto 11110_3 = 120.$$

Hence Bindy wrote down 1100 or 1010 (or 0, -1100, -1010).

3. Given ten points arranged in a regular triangular array as in the diagram, colour each point either black or white. Prove that there is always an equilateral triangle all of whose vertices have the same colour. An example of a colouring and an equilateral triangle is shown.



Solution 1:

Label the points A-J as in the diagram.

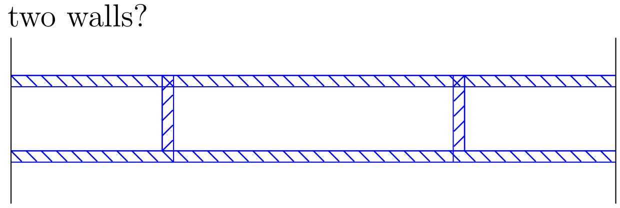
A	○
B C	• ○
D E F	D • F
G H I J	G H I J

Not all of A, G, J can be the same colour, hence at least one is a different colour to E. Then, without loss of generality, we may assume A is white and E is black. Then B and C must be different colours as otherwise one of triangles ABC or EBC will be monochromatic. Without loss of generality B is black and C is white. Then we must have D white (because BDE), F black (because ADF) and I white (because EFI), but then CDI is a white triangle.

Solution 2: Label points as in previous solution. Without loss of generality let E be black. We must have at least one of B, H, F black, without loss of generality B. Then we must have C white (because CBE), D white (because CDE) and I black (because CDI). But then we must have H white (HEI) and G white (BGI) which means GHD is entirely white.

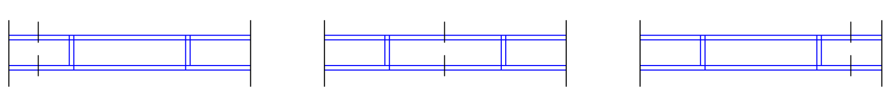
A
• ○
D • F
G H I J

4. A rope ladder connects two walls. Each of the 8 sections of the rope that make the ladder is cut with probability $\frac{1}{2}$ independently of the other sections. What is the probability that the ladder still connects the two walls?



Solution 1: Calculate the probability that the rope does not connect the two walls—we will say the ladder has fallen. The probability the ladder has fallen by two opposite horizontal sections as in figure 1 is

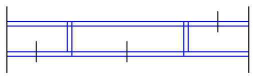
$$\frac{1}{4}+\frac{1}{4}+\frac{1}{4}-\frac{1}{4}\times\frac{1}{4}-\frac{1}{4}\times\frac{1}{4}-\frac{1}{4}\times\frac{1}{4}+\frac{1}{4}\times\frac{1}{4}\times\frac{1}{4}=\frac{37}{64}.$$

(1) 

We now consider only complementary cases to (1), i.e. at most one of each pair of horizontal opposite sections is cut such as in figure (2). The probability a ladder has fallen by the cuts in

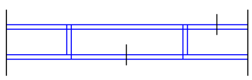
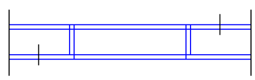
figure (2) is 2^{-7} because four more sections are forced—one cut, and three uncut. There are 8 possible ways of cutting exactly one of pair of horizontal opposite sections, with no intersection, hence the probability the ladder has fallen by such cuts is

$$4 \times \frac{1}{2^7} + 2 \times \frac{3}{4} \times \frac{1}{2^6} = \frac{7}{128}$$

(2) 

Finally we consider the cases where a pair of horizontal opposite sections is uncut (still complementary to (1)) as in figure (3). The probability a ladder has fallen by the cuts in figure (3)(a) is 2^{-7} because five more sections are forced—one cut, and four uncuts. There are four non-intersecting ways to do this. The probability a ladder has fallen by the cuts in figure (3)(b) is 2^{-8} because all sections are forced—a total of four cuts. There are two non-intersecting ways to do this. Hence the probability the ladder has fallen by such cuts is

$$4 \times 2^{-7} + 2 \times 2^{-8} = \frac{5}{128}.$$

(3)(a)  (b) 

The probability the ladder has fallen is

$$\frac{37}{64} + \frac{7}{128} + \frac{5}{128} = \frac{43}{64}$$

so the probability that the ladder still connects the two walls is

$$1 - \frac{43}{64} = \frac{21}{64}.$$

Solution 2: Split into cases based on how many of the two vertical sections are cut. If both are cut, then the probability that the rope still connects the two walls is

$$\frac{1}{4} \times \left(\frac{1}{8} + \frac{1}{8} - \frac{1}{64}\right) = \frac{15}{256}$$

since either the top horizontal section or bottom horizontal section has to stay intact. If only the left one is cut, then the probability that the remaining vertical rope is connected to the left wall is $\frac{1}{4} + \frac{1}{4} - \frac{1}{16}$, the probability that it is connected to the right wall is $\frac{1}{2} + \frac{1}{2} - \frac{1}{4}$, so the overall probability that the walls are connected is

$$\frac{1}{4} \times \left(\frac{1}{4} + \frac{1}{4} - \frac{1}{16}\right) \times \left(\frac{1}{2} + \frac{1}{2} - \frac{1}{4}\right) = \frac{21}{256}.$$

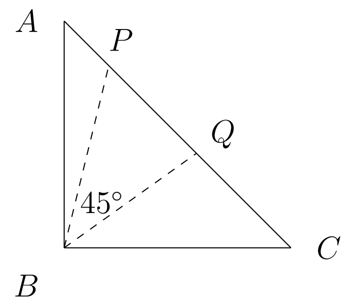
The case where only the right one is cut is similar. Finally, if neither vertical section is cut, then we just need at least one uncut rope from each pair of horizontal sections, so the probability that the walls are connected is

$$\frac{1}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{27}{256}.$$

Adding all this up:

$$\frac{15 + 21 + 21 + 27}{256} = \frac{21}{64}.$$

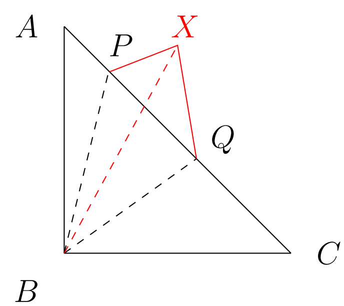
5. Consider points P and Q on the hypotenuse $\overline{AC}$ of an isosceles right-angle triangle chosen so that $\angle PBQ = 45°$ as in the figure.



Prove that the three lengths $\overline{AP}$, $\overline{PQ}$ and $\overline{QC}$ form three sides of a new right-angle triangle.

Solution 1: Reflect triangle APB about line $\overline{BP}$, and triangle CBQ about line $\overline{BQ}$. Then A and C end up at the same point X due to length $\overline{AB}$ = length $\overline{CB}$ = length $\overline{XB}$ and

$$\angle XBP + \angle XBQ = \angle ABP + \angle CBQ = 45 = \angle PBQ.$$

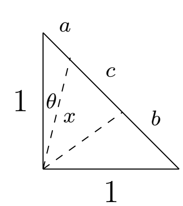


It is immediate that triangle XPQ has the relevant side lengths. Furthermore,

$$\angle PXQ = \angle PXB + \angle QXB = \angle PAB + \angle QCB = 90$$

so the triangle XPQ is right-angled.

Solution 2: Assume length $\overline{AB} = 1 =$ length $\overline{BC}$ since rescaling doesn't affect the geometry. Put $\angle PBA = \theta$, length $\overline{AP} = a$, length $\overline{PQ} = c$, length $\overline{QC} = b$ and length $\overline{BP} = x$.



Repeated applications of the sine rule give

$$a = \frac{\sin\theta}{\sin(\frac{3\pi}{4} - \theta)} = \frac{\sqrt{2}\sin\theta}{\sin\theta + \cos\theta}, \quad b = \frac{\sin(\frac{\pi}{4} - \theta)}{\sin(\frac{\pi}{2} + \theta)} = \frac{\cos\theta - \sin\theta}{\sqrt{2}\cos\theta}$$

$$c = \frac{x}{\sqrt{2}\sin(\frac{\pi}{2} - \theta)} = \frac{1}{2\sin(\frac{3\pi}{4} - \theta)\sin(\frac{\pi}{2} - \theta)} = \frac{1}{\sqrt{2}(\sin\theta + \cos\theta)\cos\theta}$$

hence the side lengths satisfy Pythagoras' theorem as required:

$$\begin{aligned} a^2+b^2 &= \frac{2\sin^2\theta}{(\sin\theta+\cos\theta)^2}+\frac{\frac{1}{2}(\cos\theta-\sin\theta)^2}{\cos^2\theta}=\frac{2\sin^2\theta\cos^2\theta+\frac{1}{2}(\cos^2\theta-\sin^2\theta)^2}{(\sin\theta+\cos\theta)^2\cos^2\theta} \\ &= \frac{\frac{1}{2}(\cos^2\theta+\sin^2\theta)^2}{(\sin\theta+\cos\theta)^2\cos^2\theta}=\frac{\frac{1}{2}}{(\sin\theta+\cos\theta)^2\cos^2\theta}=c^2. \end{aligned}$$

6. List all prime numbers contained in the sequence of 2018 numbers

$$\{101, 10101, 1010101, ..., \overbrace{101010101...1010101}^{2018\ \text{0s}}\}.$$

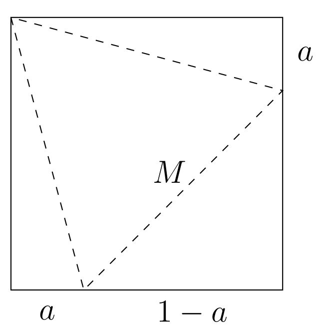
Solution: The number 101 is the only prime in the sequence.

$$101010101...1010101 = 1+10^2+10^4+...+10^{2n} = \frac{1}{99}(10^{2(n+1)}-1) = \frac{1}{99}(10^{n+1}-1)(10^{n+1}+1).$$

For $n>1$, since the two factors $10^{n+1}+1$ and $10^{n+1}-1$ are both greater 99 then they remain two non-trivial factors after dividing by factors of 99, hence the number is not prime.

7. Let r be the maximum possible quotient of the area of a square divided by the area of any equilateral triangle containing that square. Let ρ be the maximum possible quotient of the area of an equilateral triangle divided by the area of any square containing that equilateral triangle. Which of r and ρ is bigger?

Solution:* Consider an equilateral triangle of side length M inside a square which we assume has side length 1 since the quotient is independent of the scale. Simply by sliding we can see that M is maximal when the distance to the vertical equals the distance to the horizontal, both given by a in the diagram.



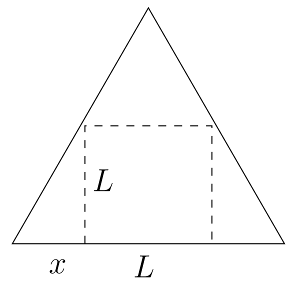
Thus

$$a^2+1=M^2=2(1-a)^2 \Rightarrow a^2-4a+1=0 \Rightarrow a=2-\sqrt{3} \Rightarrow M^2=4(2-\sqrt{3})$$

The quotient of areas is equal to the area of the equilateral triangle and given by

$$\rho = M^2\sqrt{3}/4 = 2\sqrt{3}-3.$$

Now consider a square of side length L inside an equilateral triangle which we assume has side length 1 again since the quotient is independent of the scale. Position it as in the diagram.



We have

$$L = x\sqrt{3} = \frac{(1-L)}{2}\sqrt{3} \quad \Rightarrow \quad L = 2\sqrt{3} - 3$$

The area of the square is L^2 and the area of the equilateral triangle is $\sqrt{3}/4$. This gives a lower bound for the maximum quotient of areas

$$r \geq \frac{L^2}{\sqrt{3}/4} = \frac{(2\sqrt{3}-3)^2}{\sqrt{3}/4}.$$

Hence

$$\frac{r}{\rho} \geq \frac{\frac{(2\sqrt{3}-3)^2}{\sqrt{3}/4}}{2\sqrt{3}-3} = 4(2-\sqrt{3}) > 1$$

where the last inequality uses $49 > 48 \Rightarrow 7 > 4\sqrt{3} \Rightarrow 8 - 4\sqrt{3} > 1$.

So we conclude $r > \rho$.

(* Thanks to Angelo di Pasquale for pointing out an error in the original solution.)